L 22, May 11, 2020 $\boxed{\text{Topology}}$ Unlike ~~up to now~~ so far (from now) I will follow closely LN of prof. Pultr, see his web page.

Recall the notion of <u>open</u> and <u>closed sets</u> in a metric space (X, ρ). Here $\rho: X \times X \to [0, +\infty)$ satisfies the axioms: (i) $\rho(x,y) \ge 0$ (redundant) and $= 0 \iff x = y$; (ii) $\rho(x,y) = \rho(y,x)$ (symmetry); and (iii) $x, y, z \in X \Rightarrow \rho(x,z) \le \rho(x,y) + \rho(y,z)$ ($\triangle$-inequality).

We denote, for $a \in X$ and real $r > 0$, by $B(a,r) := \{b \in X \mid \rho(a,b) < r\}$ the <u>(open) ball</u> (in X) with center a and radius r. Then $Y \subseteq X$ is an <u>open set</u> $\iff \forall a \in Y\ \exists r > 0 : B(a,r) \subseteq Y$. A set $Y \subseteq X$ is <u>closed</u> iff $X \setminus Y$ is open.

$\boxed{\text{Problem 22.1.}}$ Prove the following properties of open and closed sets in a metric space (X, ρ). (a) $\emptyset$ and X are both open and closed.

(b) If $X_1, X_2, \dots, X_n \subset X$ are open $\Rightarrow \bigcap_{i=1}^{n} X_i$ is open.

— " — closed $\Rightarrow \bigcup_{i=1}^{n} X_i$ is closed.

(c) If $X_i \subset X$, $i \in I$, are open $\Rightarrow \bigcup_{i \in I} X_i$ is open.

— " — closed $\Rightarrow \bigcap_{i \in I} X_i$ is closed.

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A <u>topology</u> or a <u>topological space</u> is a set system that has properties described in the previous problem. For more details, a <u>topology</u> on a set X is defined by <u>open sets</u> (otevřené) $\mathcal{T} \subseteq \exp(X)$ if $\mathcal{T}$ satisfies:

(ot 1) $\emptyset, X \in \mathcal{T}$; (ot 2) $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$; and (ot 3) $U_i \in \mathcal{T}$, $i \in I \Rightarrow \bigcup_{i \in I} V_i \in \mathcal{T}$.

$F \in \exp(X)$ is <u>closed</u> (uzavřená) $\Longleftrightarrow$ $X \setminus F \in \mathcal{T}$ (i.e. $X \setminus F$ is an open set).

$\boxed{\text{Problem 22.2}}$ State and prove the analogous axioms (ut 1)–(ut 3) for closed sets.

An equivalent way of defining topology on a set X is via <u>neighborhoods</u> (okolí). We say that a topology on X is introduced via n–s if for every $a \in X$ we have a set system $\mathcal{U}(a) \subseteq \exp(X)$ that is $\neq \emptyset$ and satisfies the axioms:

(ok 1) $\forall U \in \mathcal{U}(a)$: $a \in U$; (ok 2) $U, V \in \mathcal{U}(a) \Rightarrow U \cap V \in \mathcal{U}(a)$;

(ok 3) $V \subseteq X$ & $U \in \mathcal{U}(a)$ & $U \subseteq V \Rightarrow V \in \mathcal{U}(a)$; and

(ok 4) $\forall U \in \mathcal{U}(a)\ \exists W \in \mathcal{U}(a)$ s.t. $W \subseteq U$ and $\forall y \in W \Rightarrow U \in \mathcal{U}(y)$.

$\boxed{\text{Problem 22.3}}$ (ok 1)–(ok 4) imply the following: $\forall U \in \mathcal{U}(a)\ \exists W \in \mathcal{U}(a)$: $W \subseteq U$ and $\forall y \in W$ one has that $W \in \mathcal{U}(y)$. (ok 4') (You can find a proof in the LN but try to prove it by yourself.)

Now both ways of introducing a topology on X are equivalent in the sense that from n-s one can get open sets and vice versa.

neighborhoods $\rightsquigarrow$ open sets: If we have on X n-s $\mathcal{U}(a)$ satisfying (OK1) – (OK4), we define

$$\mathcal{T} := \{U \subseteq X \mid \forall a \in U : U \in \mathcal{U}(a)\}.$$

One can then check that this $\mathcal{T}$ satisfies (ot1) – (ot3).

open sets $\rightsquigarrow$ neighborhoods: If $(X, \mathcal{T})$ is a topology in the sense of (ot1) – (ot3), we define for every $a \in X$ the set system $\mathcal{U}(a) := \{U \subseteq X \mid \exists V \in \mathcal{T} : a \in V \subseteq U\}$.

Again, one can check that these set systems $\mathcal{U}(a)$, $a \in X$, satisfy axioms (OK1) – (OK4). The map $\underset{F}{\rightsquigarrow}$ is actually an involution, $F \circ F = Id$, i.e. if we apply it twice we end up at the starting place. There is yet another way of introducing topology on X, by means of the operation of closure (uzávěr), but I only describe its properties, assuming that we already have $(X, \mathcal{T})$ given by open sets.

$\boxed{\text{Definition}}$ If $\mathcal{T} \subseteq \exp(X)$ is a topology and $M \subseteq X$, we define the closure $\overline{M}$ of M as $\overline{M} := \bigcap \{F \subseteq X \mid F$ is closed and $M \subseteq F\}$. | Because of the properties of closed sets this is the same as to say that $\overline{M}$ is the $\subseteq$-smallest closed set containing M. (So, of course, $\overline{M}$ is a closed set.) (4)

Proposition Closure has the following properties (1) $M \subseteq \overline{M}$ and $\overline{\emptyset} = \emptyset$; (2) $M \subseteq N \Rightarrow \overline{M} \subseteq \overline{N}$; (3) $\overline{M \cup N} = \overline{M} \cup \overline{N}$; and (4) $\overline{\overline{M}} = \overline{M}$. | Proof. Obvious... $\boxtimes$

An analogous notion is that of an interior M^o of a set $M \subseteq X$: $M^o := \bigcup\{U \subseteq X \mid U \text{ is open and } U \subseteq M\}$. Again, it is the $\subseteq$-largest open set contained in M.

(Examples of topological spaces) We already discussed metric spaces and their open sets. A topology $(X, \mathcal{T})$ is metrisable if it can be defined ~~its~~ as open sets of a metric space (X, ρ) on X. • Discrete topology $(X, \mathcal{T})$ is $\mathcal{T} = \exp(X)$. • Indiscrete topology $(X, \mathcal{T})$ is just $\mathcal{T} = \{\emptyset, X\}$. If $|X| \geq 2$ then indiscrete topology on X is not metrisable: in any metric space (X, ρ) if $a, b \in X$ and $a \neq b$ then $\exists$ open sets $U, V \subseteq X$ s.t. $a \in U$, $b \in V$ but $U \cap V = \emptyset$ (take balls centred at a and b and with radii $< \rho(a,b)/2$) which is not possible in the ind. t. space.

For more examples (cofinite top., Alexander top., ...) see the LN.

A base of a topology $(X, \mathcal{T})$ (given by open sets) is any subset $\mathcal{B} \subseteq \mathcal{T}$ s.t. $\forall U \in \mathcal{T}: U = \bigcup\{B \in \mathcal{B} \mid B \subseteq U\}$. For example, $\{(a,b) \mid a, b \in \mathbb{Q}, a<b\}$ is a countable base of the standard, Euclidean topology on $\mathbb{R}$. A subbase of a topology $(X, \mathcal{T})$ (–‖–) is any subset $S \subseteq \mathcal{T}$ s.t. the set $\{\bigcap_{i=1}^{n} U_i \mid n \in \mathbb{N}_0, U_i \in S\}$ is a base of $\mathcal{T}$. For example, $\{(a,+\infty), (-\infty, a) \mid a \in \mathbb{Q}\}$ is a subb. of the Eucl. topology on $\mathbb{R}$.

Proposition Let X be a $(\neq \emptyset)$ set.

(a) Every $S \subseteq \exp(X)$ is a subbase of a topology $\mathcal{T}$ on X.

(b) $\mathcal{B} \subseteq \exp(X)$ is a base of a topology $\mathcal{T}$ on $X \iff X = \bigcup \mathcal{B}$ & ($U, V \in \mathcal{B} \Rightarrow U \cap V$ is a union of some elements in $\mathcal{B}$).

Proof. I will omit it, it is not hard. ☒

Example the Sorgenfrey line (see Wikipedia) According to the prev. prop., the set system $\mathcal{B} = \{[a,b) \mid a, b \in \mathbb{R}, a<b\} \subseteq \exp(\mathbb{R})$ is a base of a topology $\mathcal{T}_S$ on $\mathbb{R}$. This is the Sorgenfrey line $\mathcal{T}_S$. Not only this base is uncountable ($|\mathcal{B}| > \aleph_0$) but one can prove that $\mathcal{T}_S$ does not have any countable base — it is radically different from the Eucl. topology.

L 23 (of prof. Pultr, May 12, 2020)

Please study: Chapter V.3 Continuous maps and Chapter V.4 Basic constructions (pp. 101–105).