$\forall X \subseteq \mathbb{R}$: $X \neq \emptyset$ and is bounded from above – the set $B =$ $= \{d \in \mathbb{R} \mid b \in X \Rightarrow b \le d\}$ of the upper bounds of X is $\neq \emptyset$ – $\Rightarrow \exists\, \gamma \in B$ which is a minimum element of B, i.e. (1) $b \in B \Rightarrow \gamma \le b$. We call $\sup(X) := \gamma$ the supremum of X.

Proposition ($\exists$ of suprema) $(\mathbb{R}, <)$ is order-complete, $\forall \neq \emptyset$ and (from above) bounded set $X \subseteq \mathbb{R}$ has a supremum.

Proof. Let $X \subseteq \mathbb{R}$ be $\neq \emptyset$ and bounded from above. ~~For $b \in \mathbb{Q}$, let $s_b := (b, b, b, \dots) \in R$~~ 1. p. For every $n \in \mathbb{N}$ we define the fraction (B contains some s_b) the upper bounds of X

$$\boxed{b_n := \min(\{b \in 2^{-n}\mathbb{Z} \mid s_b \in B\})}$$

and define the sequence $x = (b_n) = (b_1, b_2, b_3, \dots)$. We claim that $x \in R$ and, in fact, $x = \sup(X)$. From the def. of b_n it follows that $\boxed{b_n \ge b_{n+1} \ge b_n - 2^{-n}}$ and from this we see that x is a Cauchy seq., so $x \in R$. Also, $x \in B$: if not then $a_n - b_n > \frac{1}{k}$ for $n \ge n_0$, for some $k \in \mathbb{N}$ and $(a_n) \in X$ – but this for n large enough that $n \ge n_0$ and $2^{-n} < \frac{1}{k}$ contradicts the def. of b_n. Also, $\forall \lambda \in B: \lambda \ge x$. Let, for a contrary, $\lambda \in B$ be such that $\lambda < x$: $b_n - c_n > \frac{1}{k}$ for $n \ge n_0$ and some $k \in \mathbb{N}$. $\lambda = (c_n)$. But this again for n large enough s.t. $n \ge n_0$ and $3 \cdot 2^{-n} < \frac{1}{k}$ contradicts the def. of b_n (for such n still $s_{b_n - 2^{-n}} \in B$). Thus $x = \sup(X)$. [$(\mathbb{Q}, <)$ is not order-complete] ∎

This completes the def. of the Dedekind order complete lin. order $(\mathbb{R}, <)$. We still have to define the field structure: the constants 0 and 1, the operations + and ·, and prove that they satisfy the axioms of an ordered field. $0 := s_0 = (0,0,\dots)$ (more precisely, the eq. block of s_0), $1 := s_1 = (1,1,\dots)$ (—"—). Both operations + and · are defined coordinate-wise: for $(a_n), (b_n) \in R$ we set $(a_n)+(b_n) := (a_n+b_n)$, $(a_n)\cdot(b_n) := (a_n \cdot b_n)$. It is easy to see that these are (binary) operations on R, which are also congruences to $\sim$, so that we in fact have two binary operations on $\mathbb{R}$. **Problem 2.1** Prove that multiplication $\cdot$ is an operation $\cdot : R \times R \to R$.

It is easy to show that + and · are associative, commutative, that 0 and 1 are their respective neutral elements, and that the distributive

$\to \left[m \geq n \Rightarrow b_n \geq b_m \geq b_n - 2^{-n} - 2^{-n-1} - \dots = b_n - 2^{1-n}. \right]$

law holds $[a\cdot(b+c) = (a\cdot b)+(a\cdot c)]$. Also, if $(a_n) \in R$ then $(-a_n) \in R$ and is an additive inverse to (a_n). **Problem 2.2** Show that if $(a_n) \in R$, $(a_n) \neq 0$ then $(1/a_n) \in R$ — what do you do if, for some $n \in \mathbb{N}$, $a_n = 0$? Also, it is easy to show that for every $(a_n), (b_n), (c_n) \in R$, $(a_n) < (b_n) \Rightarrow (a_n)+(c_n) < (b_n)+(c_n)$ and

for $(c_n) > 0$ also $(a_n) < (b_n) \Rightarrow (a_n)\cdot(c_n) <$
$< (b_n)\cdot(c_n)$. We have therefore proven the following theorem.

Theorem The above defined structure $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$ is an ordered field and $(\mathbb{R}, <)$ has the least upper bound property (A) ($\mathbb{R}$ is order-complete).

(end of insert) ~~an~~ relation R on X is preorder (quasiorder): R is refl. and trans.

Maps, mappings and functions (all are synonyma)

Two approaches. 1) Purely set-theoretical: a function f is a set such that (i) $x \in f \Rightarrow x = (a,b)$ and (ii) $(a,b), (a',b) \in f \Rightarrow a = a'$. For example, $f = \{(0,1), (1,1), (0,2)\}$. 2) More practical (which we will use): if A, B are sets then a map f from A to B, written $f : A \to B$, is a relation $f \subset A \times B$ s.t. $\forall a \in A \ \exists$ exactly one $b \in B$: $(a,b) \in f$, i.e. afb, written, as usual, $f(a) = b$.

A .... domain of f, B ... codomain of f. Thus now we have a triple (A, B, f). It is impossible to reconstruct B only from f. $B^A := \{f \mid f : A \to B\}$ is the set of all maps from A to B.

Composition of maps If $f : A \to B$ and $g : B \to C$ are maps, we define a new map $h : A \to C$ by $A \ni a \mapsto h(a) := g(f(a)) \in C$. We write $h = g \circ f$ or $h = gf$ — conflict of the order f, g and the order of reading from left to right.

$id_X : X \to X$ is the identical function, $id_X(a) = a$.

Clearly, $f(gh) = (fg)h$ (associativity) and for $f : X \to Y$ we have $id_Y \circ f = f$, $f \circ id_X = f$.

$f: X \to Y$ is __injective, an injection__ if $a, b \in X$, $a \neq b$ $\Rightarrow f(a) \neq f(b)$. It is __onto, surjective, surjection__ if $\forall b \in Y\ \exists a \in X: f(a) = b$. If f has both properties, we call it a __bijection__, or __bijective__ or __1-1 mapping__.

$\boxed{\text{Proposition}}$ $f: X \to Y$ is a bijection

$\Updownarrow$

$\exists$ a map $g: Y \to X$ s.t. $gf = id_X$ & $fg = id_Y$.

P. Easy exercise.

If $f: X \to Y$ is a map and $A \subseteq X$, we define __the image of A in f__ by $f[A] := \{f(a) \mid a \in A\} \subseteq Y$. If f is injective then the __inverse map__ to f is $f^{-1}: f[X] \to X$, $f^{-1}(b) = a \Leftrightarrow f(a) = b$.

$\boxed{\text{Insert (not in prof. Pultr's LN)}}$

$\boxed{\text{Theorem (Cantor–Bernstein)}}$ If $f: A \to B$ and $g: B \to A$ are injections, then $\exists$ a bijection $h: A \to B$ s.t. for every $a \in A$, $h(a) = f(a)$ or $h(a) = g^{-1}(a)$.

__Proof.__ Wlog $A \cap B = \emptyset$ (exercise: what to do when $A \cap B \neq \emptyset$). We consider a ~~directed~~ oriented graph $G = (V, E)$ where $V = A \cup B$, $E \subset V \times V$ and $(x, y) \in E \Leftrightarrow f(x) = y$ or $g(x) = y$. We write $x \xrightarrow{f} y$ or $x \xrightarrow{g} y$. Note that each $v \in V$ is either of __type I__: $v \longrightarrow$ (no incoming edge, one outgoing) or of type __II__: $\longrightarrow v \longrightarrow$

(one incoming and one outgoing edge, one is f-edge and the other is g-edge). Let us consider components in the unoriented graph $\overline{G}=(V,\overline{E})$, $\overline{E}=\{\{a,b\} \mid (a,b)\in E$ or $(b,a)\in E\}$. Case ① $C\subseteq V$ is a comp. containing a type I vertex $v_0\in C$. It follows ~~(by the~~ that $v_0 \to v_1 \to v_2 \to v_3 \to \cdots$ C is a one-way infinite oriented path C starting at v_0. Case ② $C\subseteq V$ is a component with no type I vertex. Let $v_0\in C$ be (type II) arbitrary. Then $v_{-1} \to v_0 \to v_1$, $v_{-1}=v_1$? If not then $v_{-2} \to v_{-1} \to v_0 \to v_1 \to v_2$, $v_{-2}=v_2$? If not then ...

Case ②a Always $v_{-i}\neq v_i$. It follows that C is a two-way infinite path: $\cdots \to \to \to \to \to \to \to \cdots$

Case ②b For some $i\in\mathbb{N}$, $v_{-i}=v_i$. It follows that then C is a directed cycle of the even length $2i$.

(L2)