

$\forall X \subseteq \mathbb{R} : X \neq \emptyset$ and X is bounded from above - the set $B = \{d \in \mathbb{R} \mid \forall x \in X \Rightarrow x \leq d\}$ of the upper bounds of X is $\neq \emptyset$ (1)
 $\Rightarrow \exists \gamma \in B$ which is a minimum element of B , i.e.

$\forall b \in B \Rightarrow \gamma \leq b$. We call $\sup(X) = \gamma$ the supremum of X .

Proposition (\exists of suprema) $(\mathbb{R}, <)$ is order-complete, $\forall \neq \emptyset$ and (from above) bounded set $X \subseteq \mathbb{R}$ has a supremum.

Proof. Let $X \subseteq \mathbb{R}$ be $\neq \emptyset$ and bounded from above. For $b \in \mathbb{Q}$, let $s_b := (b_1, b_2, b_3, \dots) \in \mathbb{R}$. For every $n \in \mathbb{N}$ we define the fraction b_n (the upper bounds of X)

$$b_n := \min \left\{ b \in 2^{-n} \mathbb{Z} \mid \exists s_b \in B \right\}$$

define the sequence $\alpha = (b_n) = (b_1, b_2, b_3, \dots)$. We claim that

$\alpha \in \mathbb{R}$ and, in fact, $\alpha = \sup(X)$. From the def. of b_n it follows that $b_n \geq b_{n+1} \geq b_n - 2^{-n}$ and from this we see that α is a Cauchy seq. $\Rightarrow \alpha \in \mathbb{R}$. Also, $\alpha \in B$ if not

then $b_n - \alpha > \frac{1}{2}$ for $n \geq n_0$, for some $n_0 \in \mathbb{N}$ and $(a_n) \in X$ - but for n large enough that $n \geq n_0$ and $2^{-n} < \frac{1}{2}$ contradicts the def. of b_n . Also, $\forall \lambda \in B : \lambda \geq \alpha$. Let, for a contrary, $\lambda \in B$ be such that $\lambda < \alpha$: $b_n - a_n > \frac{1}{2}$ for $n \geq n_0$ and some $a_n \in X$.

But this again for n large enough s.t. $n \geq n_0$ and $2^{-n} < \frac{1}{2}$ contradicts the def. of b_n (for such n still $b_n - 2^{-n} \in B$). Thus $\alpha = \sup(X)$. [$(\mathbb{Q}, <)$ is not order-complete] \square

This completes the def. of the order-complete Dedekind-complete lin. order $(\mathbb{R}, <)$. We

still have to define the field structure: the constants 0 and 1, the operations + and \cdot , and prove that they satisfy the axioms of an ordered field.

$0 := \alpha_0 = (0, 0, \dots)$ (more precisely, the eq. block of α_0), $1 := \alpha_1 = (1, 1, \dots)$ (—|—). Both operations + and \cdot are defined coordinate-wise: for $(a_n), (b_n) \in \mathbb{R}$

we set $(a_n) + (b_n) := (a_n + b_n)$, $(a_n) \cdot (b_n) := (a_n \cdot b_n)$. It is easy to see that these are operations on \mathbb{R} , which are also congruences to \sim , ^(binary) so that we in fact have two binary operations on \mathbb{R} .

Problem 2.1 Prove that multiplication \cdot is an operation $\cdot: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

It is easy to show that + and \cdot are associative, commutative, that 0 and 1 are their respective neutral elements, and that the distributive

law holds $[a \cdot (b+c) = (a \cdot b) + (a \cdot c)]$. Also, if $(a_n) \in \mathbb{R}$ then $(-a_n) \in \mathbb{R}$ and is an additive inverse to (a_n) .

$$\rightarrow [n \geq n \Rightarrow b_n \geq b_m \geq b_n - 2^{-n} - 2^{-m-1} - \dots = b_n - 2^{1-n}]$$

Problem 2.2 Show that if $(a_n) \in \mathbb{R}, (a_n) \neq 0$ then $(1/a_n) \in \mathbb{R}$ - what do you do if, for some $n \in \mathbb{N}, a_n = 0$?

Also, it is easy to show that for every $(a_n), (b_n), (c_n) \in \mathbb{R}$,

$(a_n) < (b_n) \Rightarrow (a_n) + (c_n) < (b_n) + (c_n)$ and

for $(c_n) > 0$ also $(a_n) < (b_n) \Rightarrow (a_n) \cdot (c_n) < (b_n) \cdot (c_n)$. We have therefore proven the following theorem.

Theorem The above defined

structure $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$ is an ordered field and $(\mathbb{R}, <)$ has the least upper bound property (LUB property).

(\mathbb{R} is order-complete).

(end of $(\mathbb{R}, <)$ relation ^A on \mathbb{R} is

preorder (quasiorder): R is \forall fl. and trans.

Maps, mappings and functions (all are synonymous) ④

Two approaches. ¹⁾ Purely set-theoretical: ~~a~~ a function f is a set such that (i) $x \in f \Rightarrow x = (a, b)$ and (ii) $(a, b), (a', b) \in f \Rightarrow a = a'$. For example, $f = \{(0, 1), (1, 1), (0, 2)\}$. ²⁾ More practical (which we will use): if A, B are sets then a map f from A to B , written $f: A \rightarrow B$, is a relation $f \subset A \times B$ s.t. $\forall a \in A \exists$ exactly one $b \in B$: $(a, b) \in f$, i.e. $a f b$, written, as usual, $f(a) = b$.

A ... domain of f , B ... codomain of f . Thus now we have a triple (A, B, f) . It is impossible to reconstruct f only from f :

$B^A := \{ f \mid f: A \rightarrow B \}$ is the set of all maps from A to B .

Composition of maps If $f: A \rightarrow B$ and $g: B \rightarrow C$ are maps, we define a new map $h: A \rightarrow C$ by $A \ni a \mapsto h(a) := g(f(a)) \in C$. We write $h = g \circ f$ or $h = g f$ - Conflict of the order f, g and the order of reading from left to right.

$\text{id}_X: X \rightarrow X$ is the identical function, $\text{id}_X(a) = a$.

Clearly, $f(g h) = (f g) h$ (~~also~~ associativity)

and for $f: X \rightarrow Y$ we have $\text{id}_Y \circ f = f$, $f \circ \text{id}_X = f$.

$f: X \rightarrow Y$ is injective, an injection if $a, b \in X, a \neq b \Rightarrow f(a) \neq f(b)$. It is onto, surjective, surjection if $\forall b \in Y \exists a \in X: f(a) = b$. If f has both properties, we

call it a bijection, or bijective or 1-1 mapping.

Proposition $f: X \rightarrow Y$ is a bijection



\exists a map $g: Y \rightarrow X$ s.t. $g \circ f = \text{id}_X$ & $f \circ g = \text{id}_Y$.

P. Easy exercise. If $f: X \rightarrow Y$ is a map and

$A \subseteq X$, we define the image of A in f by $f[A] := \{f(a) \mid a \in A\} \subseteq Y$. If f is injective then the inverse map to f is $f^{-1}: f[X] \rightarrow X, f^{-1}(b) = a \Leftrightarrow f(a) = b$.

What (not in prof. Putt's LN)

Theorem (Cantor-Bernstein) If $f: A \rightarrow B$ and $g: B \rightarrow A$

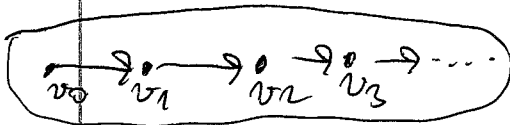
are injections, then \exists a bijection $h: A \rightarrow B$ s.t. for every $a \in A, h(a) = f(a)$ or $h(a) = g^{-1}(a)$.

Proof. $A \cap B = \emptyset$ (exercise: what to do when oriented $A \cap B \neq \emptyset$). We consider a directed graph $G = (V, E)$

where $V = A \cup B, E \subseteq V \times V$ and $(x, y) \in E \Leftrightarrow f(x) = y$ or $g(y) = x$. We write $x \xrightarrow{f} y$ or $x \xrightarrow{g} y$. Note

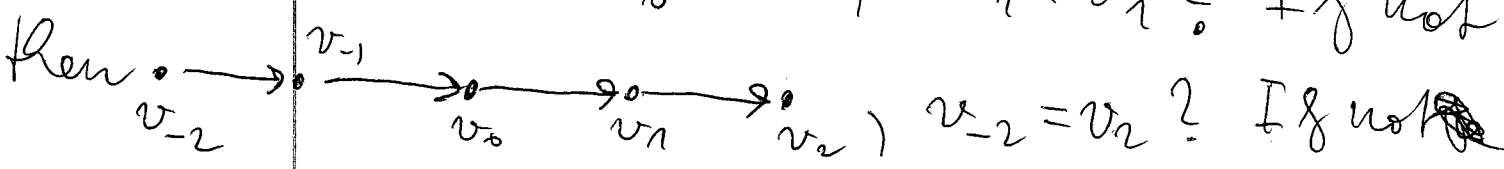
that each $v \in V$ is either of type I: $\bullet \rightarrow$ (no incoming edge, one outgoing) or of type II: $\rightarrow \bullet$ (one incoming edge, no outgoing)

(one incoming and one outgoing edge, one is f -edge and the other is g -edge). Let us consider components in the unoriented graph $\bar{G} = (V, \bar{E})$, $\bar{E} = \{ \{a, b\} \mid (a, b) \in E \text{ or } (b, a) \in E \}$. ^{Case 1} (1) $C \subseteq V$ is a comp. containing a type I vertex $v_0 \in C$. It follows that



C is a one-way infinite oriented path starting at v_0 .

^{Case 2} (2) $C \subseteq V$ is a component with no type I vertex. Let $v_0 \in C$ be arbitrary. Then $v_{-1} \rightarrow v_0 \rightarrow v_1$, $v_{-1} = v_1$? If not



Case (2a) Always $v_{-i} \neq v_i$. It follows that C is a two-way infinite path: $\dots \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \dots$

Case (2b) For some $i \in \mathbb{N}$, $v_{-i} = v_i$. It follows that C is a directed cycle of the even length $2i$.

(L2)

