[illegible] script (Ch. I) Sets, ~~rel~~ relations, maps/functions (M. Str. P.T. A)

(1) Sets: notation (1.1) $\emptyset$ ... empty set, $\in$ ... membership relation, (1.2) $A \subseteq B$ (better than $A \subset B$) means that $x \in A \Rightarrow x \in B$. We define a set by – listing its elements, e.g. $\{a\}$, $\{x, y\}$, ... [But how many elements does $\{0, a, 1, 2, \{b, \{b\}\}\}$ have?]

– by property, e.g. $\{x \mid x \text{ real } \& \ x \geq 5\}$, same as

$\{x\in\mathbb{R}\mid x\geq 5\}$. Lists, tuples: (a,b), $(x_1,x_2,\dots,x_n)$ or just $x_1,x_2,\dots,x_n$; $(x_i)_{i\in J}$. 1.3 Unions and intersections: $A\cup B$, $A\cap B$, $\bigcup_{i\in I}A_i$, $\bigcap_{i\in I}A_i$; $\bigcup\mathcal{A}:=\bigcup_{A\in\mathcal{A}}A$, $\bigcap\mathcal{B}:=\bigcap_{B\in\mathcal{B}}B$. 1.4 2-tuples or (ordered) pairs: (x,y) (similarly triples, quadruples), can model by $\in$: $(x,y):=\{\{x\},\{x,y\}\}$ (K. Kuratowski)

Exercise: check that then $(x,y)=(a,b)\iff x=a$ & $y=b$.

[Pudlák má $\{x,\{x,y\}\}$, což je trochu problematické]

Literature: P. Pudlák: Log. Found-s of Math-s & Comp. Compl-y; A Gentle Introd., Springer, Cham (2013). - On sets

Cartesian product of the sets X and Y: $X\times Y:=\{(a,b)\mid a\in X, b\in Y\}$. More gen.: the C. product of X_i, $i\in J$, is $\prod_{i\in J}X_i:=\{(x_i)_{i\in J}\mid x_i\in X_i \text{ for } \forall i\in J\}$. J finite: $X\times Y\times Z$, $D\times C$ etc. 1.5 Power set of X is the set $\mathcal{P}(X):=\{A\mid A\subseteq X\}$ (also $\exp(X)$). 1.6 Standard notation for sets of numbers: $\mathbb{N}:=\{1,2,3,\dots\}$ (also $\{0,1,2,\dots\}$) - natural numbers; $\mathbb{Z}:=\{\dots,-2,-1,0,1,2,\dots\}$ - the integers; $\mathbb{R}$ = the real numbers [and $\mathbb{Q}$!] (will define shortly). 1.7 B. Russel's paradox: $A:=\{X\mid X\notin X\}$. Then: $A\in A\rightsquigarrow$ ↯ but also $A\notin A\rightsquigarrow$ ↯.

Joke on G. Frege: Arithmetics is collapsing... More generally (and much older): Liar's paradox, The liar paradox:

S: This sentence is not true. $\rightsquigarrow$ ↯ - Theory(ies) of truth.

Nik Weaver: Truth & Assertibility, World Scientific, Singapore (2015); article in Arxiv

2 Binary relations 2.1 (binary) relation R on a set X is any subset $R\subseteq X\times X$. | $xRy:=(x,y)\in R$. $xR:=\{y\in X\mid xRy\}$,

${}_R y := \{x \in X \mid xRy\}$, $\Delta = \Delta_X := \{(x,x) \mid x \in X\}$. (diagonal relation)

2.2 <u>Composition of relations</u>: $R \circ S := \{(x,z) \in X \times X \mid \exists y \in X : xRy \,\&\, yRz\}$. <u>Inverse relation</u>: $R^{-1} := \{(x,y) \in X \times X \mid (y,x) \in R\}$. It is easy to check that:

- $R_1 \subseteq R_2,\ S_1 \subseteq S_2 \Rightarrow R_1 \circ S_1 \subseteq R_2 \circ S_2$, $R_1^{-1} \subseteq R_2^{-1}$
- $(R \circ S) \circ T = R \circ (S \circ T)$, $\Delta \circ R = R \circ \Delta = R$
- $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. Also: $\Delta \subseteq R \Rightarrow R \subseteq R \circ R$.

2.3 Some standard kinds of relations:

- $\Delta \subseteq R$ ... R is <u>reflexive</u>; - $R = R^{-1}$ ... R is <u>symmetric</u>;
- $R \circ R \subseteq R$ ... R is <u>transitive</u>; - $R \subseteq R \circ R$ ... R is <u>interpolative</u>
- R is refl. & symm. & trans. ... R is equivalence.

<u>Insert (not in A. Pultr's LN)</u>: <u>P is a partition of a set $X \neq \emptyset$</u> if: • $\bigcup P = X$, • $\emptyset \notin P$, • $A, B \in P \Rightarrow A \cap B = \emptyset$. R is an equivalence on X; $a \in X$: block (of equivalence) of a: $[a] := [a]_R := \{b \in X \mid aRb\}$.

$\boxed{\text{Proposition (eq-s} \Rightarrow \text{part-s)}}$ $X \neq \emptyset$, R is an eq. on X, P is a part. of X. The following holds.

1. $\{[a]_R \mid a \in X\} =: X/R$ is a partition of X.
2. $S \subseteq X \times X$, defined by $aSb \Leftrightarrow \exists A \in P : a, b \in A$, is an equivalence on X.

$\boxed{\text{Proposition (eq-s} \Leftrightarrow \text{part-s)}}$ ~~involution~~ $X \neq \emptyset$. $\mathcal{E} := \{\text{equivalences on } X\}$, $\mathcal{P} := \{\text{partitions of } X\}$. The maps $f : \mathcal{E} \to \mathcal{P}$ and $g : \mathcal{P} \to \mathcal{E}$ [well, we haven't yet defined maps...], $f(R) = X/R$, $g(P) = R(P)$ are inverse ~~of one another~~ mutually: $g(f(R)) = R$, $f(g(P)) = P$.

Another insert (not in A. Pultr's LN). An application of equivalence relation is a construction of $\mathbb{R}$, the real numbers (by the method of Ch. Méray, G. Cantor and E. Heine) [V. Kolman: Filosofie čísla, Akademia, Praha (2008)]. We are going to define an order-complete ordered field $\mathbb{R} = (\mathbb{R}, 0, 1, +, \cdot, <)$. We assume that the ordered field $\mathbb{Q} = (\mathbb{Q}, 0, 1, +, \cdot, <)$ has been already constructed. $(a_n) := (a_1, a_2, \dots) = (a_n)_{n \in \mathbb{N}}$ [stejně jsem ti vysvětlil axiomy usp. t. a i. usp. t.]

- $R := \{(a_n) \subseteq \mathbb{Q} \mid \forall k \in \mathbb{N}\ \exists n_0 \in \mathbb{N}: m, n \in \mathbb{N},\ m, n \geq n_0 \Rightarrow |a_m - a_n| < 1/k\}$ – the set of all Cauchy sequences of rational numbers.
- $\sim$, a relation on R: $(a_n) \sim (b_n) \Leftrightarrow \forall k \in \mathbb{N}\ \exists n_0 \in \mathbb{N}: n \geq n_0 \Rightarrow |a_n - b_n| < \frac{1}{k}$ – a_n and b_n get arbitrarily close for large n
- easy to see: $\sim$ is an equivalence on R.
- $\mathbb{R} := R/\sim \; = \{\{(a_n) \in R \mid (a_n) \sim (b_n)\} \mid (b_n) \in R\}$.
- $<$, we define a relation $<$ on R by $(a_n) < (b_n) \Leftrightarrow \exists k \in \mathbb{N}\ \exists n_0 \in \mathbb{N}$ s.t. $n \geq n_0 \Rightarrow b_n - a_n > \frac{1}{k}$. Easy to see (or exercise for you): $<$ is transitive and asymmetric, it never holds that $(a_n) < (b_n)$ & $(b_n) < (a_n)$. Law of trichotomy: if $(a_n) \not< (b_n)$ & $(b_n) \not< (a_n) \Rightarrow (a_n) \sim (b_n)$. Proof: .... Also, $<$ is a congruence w.r.t. $\sim$: $(a_n) \sim (b_n)$, $(a_n') \sim (b_n')$ then $(a_n) < (a_n') \Leftrightarrow (b_n) < (b_n')$. Thus
- $<$ is a (strict) linear ordering on $\mathbb{R}$.
- the linear order $(\mathbb{R}, <)$ is order complete:

$\forall X \subseteq \mathbb{R}: X \neq \emptyset$ & X is bounded from above — the set $B = \{d \in \mathbb{R} \mid b \in X \Rightarrow b \leq d\}$ of the upper bounds of X is $\neq \emptyset$ — $\Longrightarrow \exists \gamma \in B$ which is a minimum element of B, i.e. $b \in B \Rightarrow \gamma \leq b$. We call $\sup(X) := \gamma$ the supremum of X.

$\boxed{\text{Proposition } (\exists \text{ of suprema})}$ $(\mathbb{R}, <)$ is order-complete, $\forall \neq \emptyset$ and (from above) bounded set $X \subseteq \mathbb{R}$ has a supremum.

Proof. Let $X \subseteq \mathbb{R}$ be $\neq \emptyset$ and bounded from above. ~~For $b \in \mathbb{Q}$, let $s_b := (b, b, b, \dots) \in \mathbb{R}$~~ For every $n \in \mathbb{N}$ we define the fraction (B contains some s_b) the upper bounds of X

$$\boxed{b_n := \min(\{b \in 2^{-n}\mathbb{Z} \mid s_b \in B\})},$$

and define the sequence $\alpha = (b_n) = (b_1, b_2, b_3, \dots)$. We claim that $\alpha \in \mathbb{R}$ and, in fact, $\alpha = \sup(X)$. From the def. of b_n it follows that $\boxed{b_n \geq b_{n+1} \geq b_n - 2^{-n}}$ and from this we see that α is a Cauchy seq., so $\alpha \in \mathbb{R}$. Also, $\alpha \in B$: if not then $a_n - b_n > \frac{1}{k}$ for $n \geq n_0$, for some $k \in \mathbb{N}$ and $(a_n) \in X$ — but this for n large enough that $n \geq n_0$ and $2 \cdot 2^{-n} < \frac{1}{k}$ contradicts the def. of b_n. Also, $\forall \lambda \in B: \lambda \geq \alpha$. Let, for a contrary, $\lambda \in B$ be such that $\lambda < \alpha$: $b_n - c_n > \frac{1}{k}$ for $n \geq n_0$ and some $k \in \mathbb{N}$. (c_n) But this again for n large enough s.t. $n \geq n_0$ and $3 \cdot 2^{-n} < \frac{1}{k}$ contradicts the def. of b_n (for such n still $s_{b_n - 2^{-n}} \in B$). Thus $\alpha = \sup(X)$. $[(\mathbb{Q}, <)$ is not order-complete$]$ $\boxtimes$

This completes the def. of the order-complete Dedekind-complete lin. order $(\mathbb{R}, <)$. We