L8 (April 24, 2020) | A $\underline{\text{powersum}}$ is a formal expression $\boxed{S(x) = \sum_{i=1}^{r} p_i(x)\alpha_i^x}$, where x is a formal variable, $r \in \mathbb{N}_0$, $p_i \in \mathbb{C}[x]$ are $\neq 0$ polynomials and $\alpha_i \in \mathbb{C}$ are $\neq 0$ and mutually distinct numbers. For $r=0$ we define $S(x) := 0$. For example, $1^x + (-1)^x$ is a power sum, and so is Binet's formula $\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^x - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^x$ from the last lecture.

Every polynomial $p \in \mathbb{C}[x]$, $p \neq 0$, is a power sum $p(x)\cdot 1^x$.

Substituting $x := n \in \mathbb{N}$ we make $S(x)$ a function or sequence $S(n)$, $S: \mathbb{N} \to \mathbb{C}$.

Theorem (another $\Leftrightarrow$ for LRS) If $(a_n) = (a_1, a_2, \ldots) \subset \mathbb{C}$ is a LRS then $\exists$ a power sum $S(x)$ s.t. $\forall n \in \mathbb{N}: S(n) = a_n$. If $S(x)$ is a power sum then $S: \mathbb{N} \to \mathbb{C}$ is a LRS.

$\underline{\text{Proof.}}$ $\Rightarrow$. We assume that

$a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$ for $\forall n \in \mathbb{N}$ and some constants $k \in \mathbb{N}_0$ and $c_0, c_1, \ldots, c_{k-1} \in \mathbb{C}$, $c_0 \neq 0$. We first extend our sequence by one more term a_0 to a ~~...~~ sequence $a: \mathbb{N}_0 \to \mathbb{C}$, ~~...~~ so that we can work better with the GF of (a_n). In fact, we extend the sequence to the sequence $a: \mathbb{Z} \to \mathbb{C}$, ~~following~~ by the backward recurrence

$$a_n = -\frac{c_1}{c_0} a_{n+1} - \cdots - \frac{c_{k-1}}{c_0} a_{n+k-1} + \frac{a_{n+k}}{c_0}$$

So we have a sequence $(a_0, a_1, a_2, \ldots)$ and take its GF $\boxed{A(x) = \sum_{n=0}^{\infty} a_n x^n.}$ We know from the previous theorem that

$$A(x) = \frac{p(x)}{\underbrace{1 - x c_{k-1} - x^2 c_{k-2} - \cdots x^k c_0}_{=:q(x)}}$$

where $p, q \in \mathbb{C}[x]$ and, as is easy to see, $p = 0$ or $\deg(p) < k$. We factorize the denominator $q(x) = 1 - c_{k-1}x - c_{k-2}x^2 - \cdots - c_0 x^k$ as

$$q(x) = \prod_{i=1}^{j} (1-\alpha_i x)^{q_i}$$ where $\alpha_i \in \mathbb{C}\setminus\{0\}$ are mutually distinct and $q_i \in \mathbb{N}$, with $q_1+q_2+\dots+q_j = q$. Here the α_i are ~~the~~ roots of the char. polynomial of the recurrence. So

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{\prod_{i=1}^{j}(1-\alpha_i x)^{q_i}}$$ $p \in \mathbb{C}[x]$ with $p=0$ or $\deg(p) < q$.

We ~~par~~ expand in partial fractions[*]:

$$\dots = \sum_{i=1}^{j} \sum_{l=1}^{q_i} \frac{b_{i,l}}{(1-\alpha_i x)^l}$$ where $b_{i,l} \in \mathbb{C}$.

We have the generalized geometric series identity

$$\frac{1}{(1-\alpha x)^l} = \sum_{n=0}^{\infty} \binom{n+l-1}{n} \alpha^n x^n$$ [it follows by differentiating ~~of~~ $\frac{1}{1-\alpha x} = 1 + \alpha x + \alpha^2 x^2 + \dots$]. We substitute it in (...), compare coeff-s of x^n on both sides and get for a_n the expression

(*) Recall the precise statement and proof of the proposition on partial fractions.

$$a_n = \sum_{i=1}^{j} \sum_{l=1}^{k_i} \underbrace{\binom{n+l-1}{n}}_{B_{i,l}} \alpha_i^n,$$ which is a power sum expression $a_n = S(n)$ because $\binom{n+l-1}{n} =$
$$= \binom{n+l-1}{l-1} = \frac{(n+l-1)(n+l-2)\cdots(n+1)}{(l-1)!}$$ is a polynomial in n of degree $l-1$.

$\Leftarrow$. Let $S(t) = \sum_{i=1}^{r} p_i(t)\alpha_i^t$ be a power sum. We want to show that $(S(1), S(2), \dots)$ is a LRS. We have to make $(S(n))$ a digression to the following well known (?) lemma from linear algebra.

$\boxed{\text{Lemma (L)}}$ Let K be a field, $m, n \in \mathbb{N}$ with $m < n$, $a_{i,j} \in K$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ be mn elements of the field. Then $\exists\, \alpha_1, \alpha_2, \dots, \alpha_n \in K$, not all equal to 0_K, s.t. $\forall\, i = 1, 2, \dots, m$: $\sum_{j=1}^{n} a_{i,j}\alpha_j = 0_K$. In other words: every homogeneous linear system of equations with more unknowns than equations has a $\boxed{\text{nontrivial}}$ solution. Proof. Exercise!

Let $d := \max_{1 \le i \le r} \deg(p_i)$ and $\underline{k := r(1+d)}$.

$\forall m \in \mathbb{N}_0$ the shifted p-sum $S(t+m) =$

$$= \sum_{i=1}^{r} p_i(t+m)\,\alpha_i^{t+m} =$$

$$= \sum_{i=1}^{r} \alpha_i^m \underbrace{p_i(t+m)}\,\alpha_i^t$$

is a $\underline{\text{lin. combination}}$ (with coeff-s in $\mathbb{C}$) of the monomials $x^u \alpha_i^t$ for $0 \le u \le d$, $1 \le i \le r$ — there are $\underline{r(1+d)}$ of these monomials. Hence

[expanded by the binomial formula]

$S(t) =$ lin comb. L_0 of these monomials

$S(t+1) =$ — " — L_1 — " —

$\vdots$

$S(t+k) =$ — " — L_k — " —

By the lemma L there exist coeff-s $b_0, b_1, \dots, b_k \in \mathbb{C}$, $\underline{\text{not all } 0}$, s.t.

$$\boxed{\sum_{i=0}^{k} b_i S(t+i) \equiv 0}.$$

Let $b_{k'}$ be the first and $b_{k''}$ the last $\neq 0$ coeff. b_i; so $0 \le k' \le k'' \le k$. We define $v := k'' - k'$, $c_0 := -\dfrac{b_{k'}}{b_{k''}}$, $c_1 := -\dfrac{b_{k'+1}}{b_{k''}}$, $\dots$,

$c_{v-1} := -\frac{b_{q''-1}}{b_{q''}}$. It follows from $\boxed{\cdots}$ (6)

that for $\forall n \in \mathbb{N}$: $S(n+v) = \sum_{i=0}^{v-1} c_i S(n+i)$.

thus $(S(n))$ is a LRS. $(c_0 = -b_{q'}/b_{q''} \neq 0)$ $\boxtimes$

Lemma L can be used to show the next interesting result on LRS (which we prove next time).

Proposition Let $K \subset L$ be an extension of fields and let $(a_n) \subset K$ satisfy $a_{n+q} = \sum_{i=0}^{q-1} c_i a_{n+i}$ for $\forall n \in \mathbb{N}$ and some constants $q \in \mathbb{N}_0$, $c_0, c_1, \ldots, c_{q-1} \in L$. Then there exist $q' \in \mathbb{N}_0$ and $c_0', c_1', \ldots, c_{q'-1}' \in K$ s.t. $\forall n \in \mathbb{N}$: $a_{n+q'} = \sum_{i=0}^{q'-1} c_i' a_{n+i}$

So, for example, if $(a_n) \subset \mathbb{Q}$ is a LRS of fractions then we can assume that (a_n) is determined by a recurrence relation whose coeff-s are fractions as well. As I wrote, we prove it in the next lecture.