L5 (March 28, 2020) Proof 4. We assume for contradiction that By algebra again

$\boxed{c_n = a_1 c_{n-1} + a_2 c_{n-2} + \dots + a_k c_{n-k}}$ holds for $\forall\, n > n_0$ and some constants $a_i \in \mathbb{C}$. We know that $c_n = \frac{1}{n}\binom{2n-2}{n-1}$ and we substitute in $\boxed{\dots}$ the expressions

$$c_{n-i} = \frac{1}{n-i}\binom{2n-2i-2}{n-i-1} = \frac{1}{n-i}\cdot\frac{(2n-2i-2)!}{(n-i-1)!^2}.$$

Then we multiply the result by

$$\frac{n(n-1)\dots(n-k)\cdot(n-1)!^2}{(2n-2k-2)!}$$

to remove common factors and denominators and rearrange what we get. We use notation ($k \in \mathbb{N}_0$) $(x)_k = \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k \text{ terms}}$ where we set $(x)_0 := 1$. This is so called Pochhammer symbol. We also set ~~a_0~~ $a_0 = -1$ (as before). We get the equation

$$\sum_{i=0}^{k} a_i\cdot\widehat{(n)_{k+1}}\cdot(2n-2-2i)_{2k-2i}\cdot(n-1)_i^2 = 0$$

in which $\widehat{\dots}$ means omission of the factor $n-i$. We regard n as a formal variable and set $n = 0$. Then all summands in $\boxed{\dots}$ va-

-nish except the one for $i=0$ that is $-(-1)_k(-2)_{2k} \neq 0$. Hence $(\dots)$ is a $\neq 0$ polynomial in n with degree at most $3k$. It vanishes for every $n \in \mathbb{N}$ with $n > n_0$, which is not possible ☒

In L3 I proved the equivalence: A sequence $(a_n) = (a_0, a_1, \dots) \subset \mathbb{C}$ is holonomic $\iff$ the OGF $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is holonomic, but I did not quite justify in $\Rightarrow$ that a nontrivial recurrence (with not all coeff-s 0) yields a nontriv. lin. DE (with not all polynomial coeff-s 0), and vice versa in $\Leftarrow$. Let's do it now.

Proof. $\Longleftarrow$ (bottom of p. 2 in L3)

We know that $\sum_{i=0}^{k} q_i(x) A^{(i)}(x) = 0$ where not all $q_i \in \mathbb{C}[x]$ are 0 (and $k \in \mathbb{N}_0$). We denote $q_i(x) = \sum_{j \in J_i} c_{i,j} x^j$ with finite $J_i \subset \mathbb{N}_0$, at least one $J_i \neq \emptyset$, and all $c_{i,j} \neq 0$. Since $(\dots) = \sum_{i=0}^{k} q_i(x) \sum_{n=0}^{\infty} a_{n+i} (n+i)_i x^n$, where $(\dots)_i$ is as above (the Pochhammer symbol), we get that the coeff. of x^n in $(\dots)$ is, for every $n > n_0$,

$$\sum_{i=0}^{k} \sum_{j\in J_i} a_{n-j+i} \underbrace{c_{i,j}\,(n-j+i)_i}_{\text{polynomial in } n \text{ with degree } i \ (\text{i.e. } c_{i,j}\neq 0)}$$ (we set $u := n-j$).

Let i_0 be the maximum $i\in\{0,1,\dots,k\}$ s.t. $J_i \neq \emptyset$. We take some $j_0\in J_{i_0}$ and set $m = n-j_0+i_0$. Then the coefficient of a_m^{in} is a sum of several (an at least one) elements from $\mathbb{C}[n]$ but among these there is exactly one with the maximum degree i_0 and therefore this sum is a $\neq 0$ polynomial from $\mathbb{C}[n]$. Thus we got a nontrivial hdon. recurrence for the sequence (a_n), because

where is a $\neq 0$ poly. from $\mathbb{C}[n]$. | $\cdots + (\cdots)\,a_m + \cdots = 0$

$\Longrightarrow$ (p. 3 of L3)

Now we know, assume that for every $n > n_0$,

$$\boxed{p_0(n)\,a_n + p_1(n)\,a_{n-1} + \cdots + p_k(n)\,a_{n-k} = 0}$$ where $p_j\in\mathbb{C}[x]$ and not all of them are 0. ~~Wlog, $p_0\neq 0$.~~ On bottom of p. 3 in L3 we found that if we write $p_j(x) = \sum_{i=0}^{d_j} b_{j,i}\,(x-j)_i$ where d_j is $\deg p_j$ if

$p_j \neq 0$ and -1 if $p_j = 0$ (the Σ is then 0) and $b_{j,i} \in \mathbb{C}$ (we use the basis $\{(x-j)_i \mid i \in \mathbb{N}_0\}$ of $\mathbb{C}[x]$ instead of $\{x^i \mid i \in \mathbb{N}_0\}$) then for $j = 0, 1, \dots, k$ the term $p_j(n)\,a_{n-j}$ in $\underbrace{\dots\dots}$ is the coefficient of x^n in

$$\sum_{i=0}^{d_j} b_{j,i}\, x^{i+j} A^{(i)}(x).$$

The sum $\underbrace{\dots}$ is 0, which means that

$$[x^n] \sum_{j=0}^{k} \sum_{i=0}^{d_j} b_{j,i}\, x^{i+j} A^{(i)}(x) = 0$$

for every $n > n_0$, where the notation $[x^n]\,F(x)$ (in) denotes the coefficient of x^n in $F(x)$. To make the equality holds for $\forall n \in \mathbb{N}_0$ (and remove the finitely many possible exceptions), we add above an appropriate polynomial $q(x) \in \mathbb{C}[x]$ and get that, in the ring $\mathbb{C}[[x]]$ of formal power series,

$$\boxed{q(x) + \sum_{j=0}^{k} \sum_{i=0}^{d_j} b_{j,i}\, x^{i+j} A^{(i)}(x) = 0.}$$

Now we need to show that for some i the accumulated polynomial coeff. of $A^{(i)}(x)$ in $\boxed{\cdots}$ is

$\neq 0$. Let ~~[illegible]~~ $j_0 \in \{0,1,\dots,k\}$ be such ~~the exact~~

$p_{j_0}(x) \neq 0$ and let i_0 be s.t. $b_{j_0 i_0} \neq 0$. Then the polynomial that is the coeff. of ~~[illegible]~~ $A^{(i_0)}(x)$ in $\boxed{\cdots}$

is $\neq 0$ because it has at least ~~[illegible]~~ the $\neq 0$ monomial $b_{j_0 i_0} x^{i_0+j_0}$. Thus we indeed get a linear

diff. eq. ~~for A(x)~~ with polynomial coefficients $q(x), q_0(x), q_1(x), \dots, q_\ell(x)$, as at the top of p. 4 of L3, not all of which are 0 polynomials. $\boxtimes$

Exercise: what is the order ℓ of this equation, in terms of the original holonomic recurrence $\boxed{\cdots}$ on p. 3 (of this lecture)?

Uff... this was longer than I expected, and so I conclude my lecture here, with the mention of the book ~~R.P.~~ Enumerative Combinatorics, Volume 2 by R.P. Stanley. In Chapter 6 it treats D-finite (i.e. holonomic) power series.