(L3) (1st Coronavirus lecture) In this lecture we generalize the recurrence $C_n = \frac{4n-6}{n} C_{n-1}$ for the Catalan numbers, and we also generalize the method of obtaining this recurrence from the algebraic equation for the OGF. A sequence $(a_n) = (a_0, a_1, \dots) \subset \mathbb{C}$ is **holonomic**, or **P-recursive**, if for some $n_0 \in \mathbb{N}$, k and some polynomials $p_0, \dots, p_k \in \mathbb{C}[x]$, $p_0 \neq 0$, we have for every $n \geq n_0$ that (for $k=0$ the RHS is $:= 0$)

$$p_0(n)\, a_n = p_1(n) a_{n-1} + \dots + p_k(n)\, a_{n-k}.$$

In other words, $a_n = \frac{p_1(n)}{p_0(n)} a_{n-1} + \dots + \frac{p_k(n)}{p_0(n)} a_{n-k}$.

For example, (C_n) is holonomic. A power series $F = F(x) = \sum_{n=0}^{\infty} f_n x^n \in \mathbb{C}[[x]]$ is (formal - we do not care about the convergence) **D-finite** or **holonomic** if

$$\dim_{\mathbb{C}(x)} (\{F^{(n)}(x) \mid n \in \mathbb{N}_0\}) < \infty,$$

- the set of derivatives of F has finite (linear) dimension over the field $\mathbb{C}(x)$ of rational functions (ratios of polynomials). Explicitly:

$\exists\, k \in \mathbb{N}_0$ and polynomials $q_0, q_1, \dots, q_k \in \mathbb{C}[x]$, not all zero, s.t. $q_0 F + q_1 F' + \dots + q_k F^{(k)} \equiv 0$. Exercise

$\Longleftrightarrow$ $\exists$ polynomials $q, q_0, \dots, q_k \in \mathbb{C}[x]$, not all 0, s.t.

$$q + q_0 F + q_1 F' + \dots + q_k F^{(k)} = 0$$

$\boxed{\text{Theorem}}$

A sequence $(a_n) = (a_0, a_1, \dots) \subset \mathbb{C}$ is holonomic

$$\Updownarrow$$

Its OGF $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is holonomic.

$\boxed{\text{Proof.}}$ $\Uparrow$ – suppose that $A(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathbb{C}[[x]]$ is a holonomic ~~seq~~ power series. Then

$$\boxed{\sum_{i=0}^{k} q_i(x) \cdot A^{(i)}(x) = 0}$$

for some $q_i \in \mathbb{C}[x]$, not all 0. Since $A^{(i)}(x) = \sum_{n=0}^{\infty} a_n \, n(n-1)\cdots(n-i+1)\, x^{n-i} =$

$$= \sum_{n=0}^{\infty} \underbrace{a_{n+i}(n+i)(n+i-1)\cdots(n+1)}_{\in \mathbb{C}[n]} x^n,$$

equating the coeff. of x^n, $n > n_0$, on the LHS of $\boxed{\dots}$ to 0 we get for (a_n) a holonomic recurrence.

Exercise How large should be n_0 here?

$\Downarrow$ Suppose that $(a_n)=(a_0,a_1,\dots)\subset\mathbb{C}$ is a holonomic sequence: $\boxed{p_0(n)a_n=p_1(n)a_{n-1}+\dots+p_q(n)a_{n-q},\ n>n_0}$, 3
$p_i(x)\in\mathbb{C}[x]$
and $p_0(x)\neq 0$. We denote by $(x)_i$, $i\in\mathbb{N}_0$, the polynomial $x(x-1)\cdots(x-i+1)$, $(x)_0:=1$. The $\mathbb{C}$-vector space of polynomials $\mathbb{C}[x]$ has the canonic basis $\{x^i\mid i\in\mathbb{N}_0\}=\{1,x,x^2,\dots\}$, but $\{(x)_i\mid i\in\mathbb{N}_0\}=\{1,x,x(x-1),\dots\}$ is another basis. As we know

$$A^{(i)}(x)=\left(\sum_{n=0}^{\infty}a_nx^n\right)^{(i)}=\sum_{n=0}^{\infty}a_n(n)_i\,x^{n-i}=\sum_{n=0}^{\infty}a_n(n)_i\,x^{n-i}$$

(because $(x)_i$ has roots $0,1,\dots,i-1$). If $\deg(p_0)=:d\in\mathbb{N}_0$, we have the expression $p_0(x)=\sum_{i=0}^{d}b_i(x)_i$ for some $b_i\in\mathbb{C}$. Thus $p_0(n)a_n$ (in $\boxed{\dots}$) is the coeff. of x^n in

$$\sum_{i=0}^{d}b_ix^iA^{(i)}(x).$$

Similarly, for $i=0,1,\dots$ we have $A^{(i)}(x)=\sum_{n=1}^{\infty}a_{n-1}(n-1)_i\,x^{n-1-i}$. We write $p_1(x)=\sum_{i=0}^{d'}b'_i(x-1)_i$, $b'_i\in\mathbb{C}$, and get that $p_1(n)a_{n-1}$ (in $\boxed{\dots}$) is the coeff. of x^n in $\sum_{i=0}^{d'}b'_ix^{i+1}A^{(i)}(x)$. And so on.

We therefore see that for some polynomials $q, q_0, \dots, q_\ell \in \mathbb{C}[x]$, the linear combination

$q(x) + q_0(x)A(x) + q_1(x)A'(x) + \dots + q_\ell(x)A^{(\ell)}(x)$ is $\equiv 0$ and we have ~~the~~ ~~for~~ a diff. equation for A showing that $A(x)$ is holonomic ~~[illegible]~~ $\boxtimes$

Remark. The $q(x)$ takes care of the $n > n_0$ ~~[illegible]~~ condition. Also, I neglected to justify in both $\Uparrow$ and $\Downarrow$ that the resulting recurrence, resp. diff. equation, is non-trivial (not all coeff-s are 0). This is an exercise for you, and I will return to this point next time.

Theorem If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is algebraic — $P(x, A(x)) \equiv 0$ for a nonzero $P \in \mathbb{C}[x, y]$ — then $A(x)$ is holonomic.

Proof. So, by the assumption,

$$p_0 A^d + \frac{p_1}{\cancel{p_0}} A^{d-1} + \dots + \frac{p_{d-1}}{\cancel{p_0}} A + \frac{p_d}{\cancel{p_0}} \equiv 0 \quad (*)$$

for some $d \in \mathbb{N}$ and $p_i \in \mathbb{C}[x]$, $p_0 \neq 0$. Taking a derivative of $(*)$ ($p_0 A^d \rightsquigarrow p_0' A^d + p_0 d A^{d-1} A', \dots$) we express A' as

$$A' = \frac{q(1,0) + q(1,1)A + \dots + q(1,d-1)A^{d-1}}{r(1,0) + r(1,1)A + \dots + r(1,d-1)A^{d-1}},$$

where

$q(\cdot,\cdot), r(\cdot,\cdot) \in \mathbb{C}[x]$; we have this expression also for $A^{(0)} = A$. Taking derivatives repeatedly we have – using the replacement $A^d = \frac{p_1 A^{d-1} + \dots + p_{d-1}A + p_d}{p_0}$ – that for any $j = 0, 1, 2, \dots,$

$$A^{(j)} = \frac{q(j,0) + q(j,1)A + \dots + q(j,d-1)A^{d-1}}{r(j,0) + r(j,1)A + \dots + r(j,d-1)A^{d-1}}$$

where again $q(\cdot,\cdot), r(\cdot,\cdot) \in \mathbb{C}[x]$. We take $d+1$ of these derivatives: $\underset{\substack{\| \\ A}}{A^{(0)}} = \frac{N_0}{D_0}$, $A' = \frac{N_1}{D_1}$, $A'' = \frac{N_2}{D_2}, \dots, A^{(d)} = \frac{N_d}{D_d}$.

Exercise We may assume that the denominators are equal, $D_0 = D_1 = D_2 = \dots = D_d$. Since each numerator $N_\cdot$ is determined by a d-tuple from $\mathbb{C}[x]^d$, it follows (by the exercise below) that there exist rat. functions $s_0, s_1, \dots, s_d \in \mathbb{C}(x)$, not all 0, s.t. $s_0 N_0 + s_1 N_1 + \dots + s_d N_d \equiv 0$. Thus we have a nontrivial ~~holonomic~~ diff. equation $s_0 A + s_1 A' + s_2 A'' + \dots + s_d A^{(d)} \equiv 0$ showing that A is holonomic.

$\boxtimes$

We used the next result whose proof is left for you as an exercise. ⑥

$\boxed{\text{Lemma}}$ (but it is more important than that) If F is a field and

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \dots + a_{1,m}x_m &= 0_F \\ a_{2,1}x_1 + a_{2,2}x_2 + \dots + a_{2,m}x_m &= 0_F \\ &\vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \dots + a_{n,m}x_m &= 0_F \end{aligned}$$

$(m, n \in \mathbb{N})$ $(a_{\cdot,\cdot} \in F)$

is a homogeneous lin. system with m unknowns, n equations and $\underline{m > n}$, then it has a non trivial solution $x_1, \dots, x_m \in F$ (not all x_i are 0_F).