(L 10), May 8, 2020 | To continue the last remarks, all proofs of the SML theorem known use p-adic numbers $\mathbb{Q}_p$ in some form. Because of the equivalence we proved (LRS $\Leftrightarrow$ power sums), this theorem is equivalently about the LRS. Its another, more popular and equivalent, statement is as follows. (1)

Theorem (the SML theorem again) If $A = (a_n) \subset \mathbb{C}$ (or $\subset K$ for a field K with $\mathrm{char}(K) = 0$) is a sequence satisfying for $\forall n \in \mathbb{N}$ the recurrence ((LRS)) $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$ $(c_0 \neq 0_K)$ for some constants $k \in \mathbb{N}_0$ and $c_i \in \mathbb{C}$ (or $\in K$ as in), then there $\exists$ a modulus $m \in \mathbb{N}$ s.t. for every $j = 1, 2, \dots, m$ the zero set (of A in $n \equiv j \pmod m$)

$$Z_{A,j} := \{n \in \mathbb{N}_0 \mid a_{mn+j} = 0\}$$

either equals $\mathbb{N}_0$ or is finite. This applies to any LRS, not just on the non-degenerate ones. For example, the LRS $A=(a_n)=(a_1,a_2,\dots)=(0,1,0,1,0,1,\dots)$ $[a_{n+2}=0\cdot a_{n+1}+1\cdot a_n]$ has, for $m=2$ and, $Z_{A,1}=\mathbb{N}_0$ and $Z_{A,2}=\emptyset$. $\boxed{\text{End of the remarks}}$

Another important, interesting and useful result related to the LRS and rational GFs is the transfer matrix method, which is well known also to ~~the~~ physicists. I will begin with seemingly purely algebraic result and then show its application to counting weighted ~~directed~~ walks in directed multigraphs.

We consider an arbitrary ring R (commutative and with 1_R) and the ring $R[[x]]$ of formal power series with coeff-s in R. If $A\in R^{n\times n}$ is an $n\times n$ matrix with entries in R, i.e. $A:[n]^2\to R$, we denote by $A_{i,j}$ the i,j-entry, i.e. $A(i,j)$. Recall the (in general non-commutative) operation of matrix multiplication: if $A \in R^{m \times n}$, $B \in R^{n \times p}$ then $C := AB \in R^{m \times p}$ and $C_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$.

$\boxed{\text{Theorem (the TMM, algebraic part)}}$ If $A \in R^{n \times n}$ and $i, j \in [n] = \{1, 2, \dots, n\}$, then we have in $R[[x]]$ the identity

$$\sum_{k=0}^{\infty} (A^k)_{i,j} x^k = \frac{(-1)^{i+j} \det((I - xA)[j,i])}{\det(I - xA)}$$

Here I is the unit $n \times n$ matrix in $R^{n \times n}$, so

$$I = \begin{pmatrix} 1_R & & 0_R \\ & 1_R & \\ 0_R & & 1_R \end{pmatrix},$$

xA is the matrix obtained from A by multiplying each entry of it by x, and $-$ is term-wise subtraction. Thus, if $A = (a_{i,j})_{i,j=1}^{n} \in R^{n \times n}$, then $I - xA = (\delta_{i,j} - a_{i,j} x)_{i,j=1}^{n}$ where $\delta_{i,j}$ is Kronecker's δ in R:

$$\delta_{i,j} = \begin{cases} 0_R & \dots\ i \neq j \\ 1_R & \dots\ i = j. \end{cases}$$

Further, if $B \in R^{n \times n}$ and $i, j \in [n]$ then $B[i,j]$ denotes the matrix (in $R^{(n-1)\times(n-1)}$) obtained from B by deleting from B the i-th row and j-th column.

The numerator and the denominator in the identity [det is of course the determinant, the well known map $\det: R^{n\times n} \to R$] are polynomials in $R[x]$, here understood as $\subset R⟦x⟧$, and the denominator is a unit in $R⟦x⟧$ because $\det(I - xA) = 1_R + \bullet x + \cdots + \bullet x^n$ (where $\bullet \in R$), thus we can divide by it in $R⟦x⟧$ and the right side of the identity is well defined. Now let us have a look at the $\boxed{\text{proof}}$ of this identity. A physicist(*) would compute:

$$\left(\sum_{k=0}^{\infty} (A^k)_{ij} x^k\right)_{i,j=1}^{n} = \sum_{k=0}^{\infty} A^k x^k$$

$$= (Ix^0 - Ax)^{-1}$$

$$= (I - xA)^{-1} =$$

(*) I admire the style ~~of~~ computations are done in physics but still am a mathematician.

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$$= \left(\frac{(-1)^{i+j} \det((I-xA)[j,i])}{\det(I-xA)} \right)_{i,j=1}^{n}$$ where in the

last = we have used the formula from the linear algebra for the inverse matrix to a given regular matrix, here the matrix $I-xA = (\delta_{i,j} - a_{i,j}x)_{i,j=1}^{n}$. If we compare the entries in the initial and final $n \times n$ matrix, we get the stated identity. $\boxtimes$

I ~~attempt~~ now ~~to~~ explain this computation from the perspective of a pedantic algebraist. We are working here in two very similar but really ~~distinct~~ different rings,

$$U := (R[[x]])^{n \times n} \text{ and}$$

$$V := (R^{n \times n})[[x]]. \text{ So } U$$

is the ring of n by n matrices whose entries are formal power series with coeff-s in R and V is the ring of formal power series with coeff-s

in the ring of $n \times n$ matrices with entries in R.
These are two obviously related but as types completely different classes of mathematical/algebraic objects/structures. The equalities in the above computation jump between U and V in this way:

$$U = V = V = U = U.$$

So only the 2nd and 4th $=$ equate objects of the same types, the 1st and 3rd are not really equalities but isomorphisms between the rings U and V. I have not seen in the literature an account on the TOIM that would explain this point clearly (most are even unaware of it), ~~attempted to~~ I wrote such myself (google it) and I attempt to do it again here. But algebraic niceties aside, the combinatorial application(s) is (are) more important.

To be continued ... MM