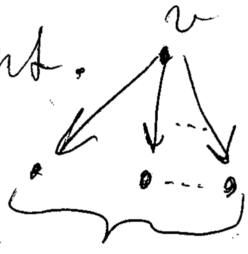
Combinatorial Counting (L1)

Enumerative Combinatorics – discipline of Comb-cs concerned with counting things, with determination of cardinalities of finite sets, often (but not only) by means of generating functions.

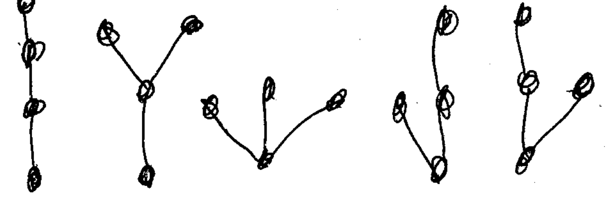
Example: The Catalan #s

In fact: The up trees An *up-tree* $T=(r,V,E,L)$ is a structure such that (r,V,E) is a *finite rooted tree* [V are the *vertices*, $E\subset V\times V$ are the *edges*, $r\in V$ is the *root*, and $\forall v\in V\ \exists$ unique r-v *walk*, a finite sequence $P=v_0v_1\dots v_k$ s.t. $v_0=r$, $(v_{i-1},v_i)\in E$, and $v_k=v$. In fact, P is always a *path*: $i\neq j \Rightarrow v_i\neq v_j$. Also, T has no loops, each $v\neq r$ has exactly one parent $\bullet\longrightarrow\bullet\, v$, r has no parent. A vertex with no child is *a leaf*.]

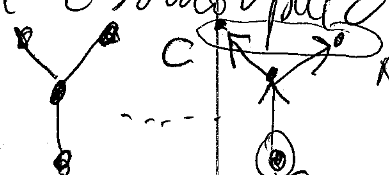


L is a set of *linear orders* $L=\{L_v \mid v\in V\}$ on the sets L_v of children of the vertices $v\in V$.

More intuitively and visually



are all non-isomorphic up trees with $|V|=4$ vertices.

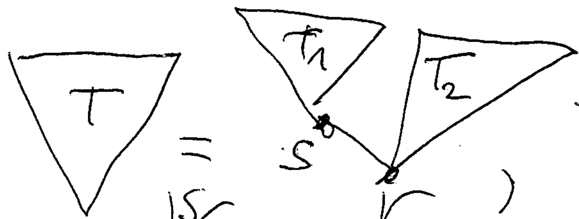


left-right order = the order on C in L

Let $\mathcal{T}$ be the set of all distinct (i.e. $\mathcal{T}$ is the set of all classes of isomorphism of up-trees) up trees. Let $\mathcal{T}_n = \{ T \in \mathcal{T} \mid |T| := |V| = n \}$ be trees with n vertices, and $c_n = |\mathcal{T}_n|$. $\boxed{\text{Question}}$ of E.C.: $\boxed{c_n = ?}$ (= the Catalan numbers). We have $c_1 = 1$ (•), $c_2 = 1$, $c_3 = 2$ and, as we have seen, $c_4 = 5$. How does the sequence $(1, 1, 2, 5, \dots)$ continue further? We will see.

$\boxed{\text{Proposition}}$ $\exists$ a bijection $f : \mathcal{T} \setminus \mathcal{T}_1 \to \mathcal{T} \times \mathcal{T}$, $T \mapsto (T_1, T_2)$, s.t. $|T| = |T_1| + |T_2|$.

Proof. $T \neq \bullet \Rightarrow$



, T_1 = the descendants of the 1st child of the root (s included) and T_2 = the rest of T. It is easy to check that this is the desired bijection (exercise for you). ⊠

Restricting f to $\mathcal{T}_n$, $n \geq 2$, we get bijections $\mathcal{T}_n \leftrightarrow \bigcup_{k=1}^{n-1} \mathcal{T}_k \times \mathcal{T}_{n-k}$, and thus the basic recurrence

$$\boxed{c_1 = 1 \text{ and } c_n = \sum_{k=1}^{n-1} c_k c_{n-k} \text{ for } n \geq 2.}$$

By this recurrence we can compute as many Catalan #s as we wish:

$(c_n) = (c_n)_{n\geq 1} = (c_1, c_2, c_3, \dots)$ See OEIS... integ. (On-line encyclopedia of seq.)

$= (1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \dots$

$\boxed{\text{Can you guess } \cancel{\text{them}} \text{ which } c_n \text{ are odd numbers?}}$
$\boxed{\text{(– without reading further –)}}$

$\boxed{\text{Proposition}}$ The Catalan number c_n is odd $\iff n = 2^m$, $m \in \mathbb{N}_0$.

[$\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$] Proof. By induction using the basic recurrence for the c_n. For $n=1$, $c_1 = 1$ ✓. If $n > 1$ and is even then

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k} = c_{n/2}^2 + 2\sum_{k=1}^{\frac{n}{2}-1} c_k c_{n-k} \equiv c_{n/2} \pmod 2.$$

If $n > 1$ and is odd then

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k} = 2\sum_{k=1}^{(n-1)/2} c_k c_{n-k} \equiv 0 \pmod 2.$$

Thus for $n = 2^k m$ with $k \in \mathbb{N}_0$ and $m \in \mathbb{N}$ an odd number we have that $c_n \equiv c_m \begin{cases} \equiv 0 & \text{for } m > 1 \\ \equiv 1 & \text{for } m = 1 \end{cases}$ ⊠

There are quite a few articles on modular behavior of the Catalan numbers, and it can be effectively described for any modulus m. More about this, and in a greater generality, later. (maybe)

How fast do c_n grow? We deduce it, at least approximately, from the basic recurrence. For $n \geq 3$ it gives that $c_n \geq 2c_1 c_{n-1} = 2c_{n-1}$. Since $c_7 = 132 > 2^7 = 128$, we get by induction that $\boxed{c_n > 2^n \text{ for every } n \geq 7.}$ It is a little harder to obtain from the basic r. an exponential upper bound on c_n. We need a lemma for it.

Lemma For every $n = 2, 3, \dots$,
$$\sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} < \frac{8}{n^2}$$

Proof. As $(k(n-k))^{-1} = n^{-1}(k^{-1} + (n-k)^{-1})$,
$$\sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} = \frac{2}{n^2}\sum_{k=1}^{n-1}\frac{1}{k^2} + \frac{4}{n^3}\sum_{k=1}^{n-1}\frac{1}{k} < \frac{4}{n^2} + \frac{4}{n^2} = \frac{8}{n^2}$$
because
$$\sum_{k=1}^{n-1}\frac{1}{k^2} < 1 + \sum_{k=2}^{\infty}\frac{1}{k(k-1)} = 1 + \sum_{k=2}^{\infty}\left(\frac{1}{k-1} - \frac{1}{k}\right) = 2.$$
$\boxtimes$

Proposition (exp. upper bound on c_n) For $\forall n \in \mathbb{N}$: $c_n \leq \frac{8^{n-1}}{n^2} < 8^n$.

Proof. We seek an upper bound in the form $c_n \leq c\frac{\alpha^n}{n^2}$ where $c > 0$ and $\alpha > 1$ are constants, which are to be determined. Suppose it holds for c_q for every $q < n$. Then the basic r. and the lemma give for $n \geq 2$ that

$$c_n = \sum_{q=1}^{n-1} c_q c_{n-q} \leq \sum_{q=1}^{n-1} \frac{c d^q}{q^2} \cdot \frac{c d^{n-q}}{(n-q)^2} < \frac{8c^2 d^n}{n^2}.$$

This is $\leq c\frac{d^n}{n^2}$ iff $8c^2 \leq c \iff c \leq \frac{1}{8}$. We set $c = \frac{1}{8}$.

To start the induction, we need that $c_n \leq c\frac{d^n}{n^2} = \frac{d^n}{8n^2}$ holds for $n = 1$: $\frac{d^1}{8 \cdot 1^2} \geq c_1 = 1 \iff d \geq 8$.

Thus $c_n \leq \frac{d^n}{8n^2} = \frac{8^{n-1}}{n^2}$ for every $n \in \mathbb{N}$. — we set $d = 8$. ⊠

Generating function and formula for Catalan numbers

$$C(x) := \sum_{T \in \mathcal{T}} x^{|T|} = \sum_{n=1}^{\infty} |\mathcal{T}_n| x^n = \sum_{n=1}^{\infty} c_n x^n.$$

By the basic r. we have:

$$C(x)^2 = \sum_{n=1}^{\infty} c_n x^n \sum_{n=1}^{\infty} c_n x^n = \sum_{n=2}^{\infty} \left(\sum_{q=1}^{n-1} c_q c_{n-q}\right) x^n = \sum_{n=2}^{\infty} c_n x^n = C(x) - x.$$

Hence

$$\boxed{C(x)^2 - C(x) + x = 0.} \quad \left(x \in \mathbb{C}, |x| < \tfrac{1}{8}\right).$$

(6) Solving the quadratic eq. we get that $C(x) =$
$= \frac{1}{2}(1 \pm \sqrt{1-4x})$ but what is $\sqrt{1-4x}$? I. Newton:

for any $\alpha \in \mathbb{R}$ and $x \in \mathbb{C}$, $|x|<1$, $(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$
where $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}$.

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \dots$$

Thus the correct sign in is $-$ (because $C(x)$ has no constant term) and $\boxed{C(x) = \frac{1-\sqrt{1-4x}}{2}}$ and from $\sqrt{1-4x} = \sum_{n=0}^{\infty} \binom{1/2}{n}(-4x)^n$ we get that the n-th Catalan number is $\boxed{C_n = \left(-\frac{1}{2}\right)\binom{1/2}{n}(-4)^n}$. — to be continued!

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