Proposition 6.4.1. *Suppose that $r > 0$ is a real number, $I \supset [0, r]$ is an open interval, $f : I \to \mathbb{R}$ is a function satisfying on I the differential equation*

$$f = \alpha f'', \ \alpha \in \mathbb{R}\backslash\{0\},$$

and $f(0) \neq -f(r)$ or $f'(0) \neq f'(r)$. Then at least one of the four numbers

$$\alpha, c_1 = rf(0) + rf(r), \ c_2 = f'(r) - f'(0), \ \text{ and } \ r^2$$

is irrational.

Proof. For $n \in \mathbb{N}_0$, let $f_n(x) = x^n(r-x)^n/n!$ and

$$I_n = \int_0^r f_n(x)f(x)\,dx = \frac{1}{n!}\int_0^r x^n(r-x)^n f(x)\,dx\,.$$

Then, since $(\alpha f')' = f$ and, for $n \geq 1$, $f_n(0) = f_n(r) = 0$, we have $I_0/\alpha = f'(r) - f'(0)$ and $I_1/\alpha = -\int_0^r f_1' f' = -r\int_0^r f' + 2\int_0^r xf' = r(f(0) + f(r)) - 2\alpha(f'(r) - f'(0))$. Thus

$$I_0 = \alpha c_2 \ \text{ and } \ I_1 = \alpha c_1 - 2\alpha^2 c_2\,.$$

For $n \geq 2$ we have

$$\begin{aligned} f_n''(x) &= \frac{x^{n-2}(r-x)^n + x^n(r-x)^{n-2}}{(n-2)!} - \frac{2nx^{n-1}(r-x)^{n-1}}{(n-1)!} \\ &= \frac{x^{n-2}(r-x)^{n-2}(r^2 - 2x(r-x))}{(n-2)!} - \frac{2nx^{n-1}(r-x)^{n-1}}{(n-1)!} \\ &= r^2 f_{n-2}(x) - (4n-2)f_{n-1}(x) \end{aligned}$$

and, integrating by parts and using the above relations and $f_n'(0) = f_n'(r) = 0$,

$$\begin{aligned} I_n &= \int_0^r f_n(x)f(x)\,dx = -\alpha\int_0^r f_n'(x)f'(x)\,dx = \alpha\int_0^r f_n''(x)f(x)\,dx \\ &= \alpha(2-4n)I_{n-1} + \alpha r^2 I_{n-2}\,. \end{aligned}$$

It follows by induction on n that I_n is a polynomial in $\mathbb{Z}[\alpha, c_1, c_2, r^2]$ with degree at most $n+2$. For any $\delta > 0$ we have

$$|I_n| < \delta^n \text{ for } n > n_0, \text{ and } I_n \neq 0 \text{ for infinitely many } n\,.$$

The upper bound is clear because by the definition $I_n \ll r^{2n}/n!$. If $I_n = 0$ for every $n > n_0$, or even if $I_n = I_{n+1} = 0$, then by running the recurrence for I_n backwards (which we can as $\alpha r^2 \neq 0$ is constant) we deduce that $I_n = 0$ for every $n \in \mathbb{N}_0$, in contrary with the assumption that $c_1 \neq 0$ or $c_2 \neq 0$ and hence $I_0 \neq 0$ or $I_1 \neq 0$.

Now assume to the contrary that all four numbers α, c_1, c_2, and r^2 are rational. Let $a \in \mathbb{N}$ be their common denominator. Then $a^{n+2}I_n \in \mathbb{Z}$ for every $n \in \mathbb{N}_0$ and is nonzero for infinitely many n, in particular $|a^{n+2}I_n| \geq 1$ for infinitely many n. At the same time $a^{n+2}I_n \to 0$ as $n \to \infty$, which is a contradiction. □

Corollary 6.4.2 (irrationality of $e^{a/b}$ and π^2). *For every nonzero $r \in \mathbb{Q}$, the number e^r is irrational. Also, the number π^2 is irrational. In particular, the numbers e and π are irrational.*

Proof. We first apply Proposition 6.4.1 to rational $r > 0$ and function $f(x) = e^x$; for $r < 0$ we use that $e^r = 1/e^{-r}$. As $e^x = (e^x)' = (e^x)''$, we have $\alpha = 1$. Since $\alpha, r, r^2 \in \mathbb{Q}$, $c_1 = r + re^r > 0$ and $c_2 = e^r - 1$, we conclude that e^r is irrational. Then we apply Proposition 6.4.1 to $r = \pi$ and function $f(x) = \sin x$. As $(\sin x)' = \cos x$ and $(\sin x)'' = -\sin x$, we have $\alpha = -1$. Now $\alpha = -1$, $c_1 = 0$ and $c_2 = -2$, and we conclude that $r^2 = \pi^2$ is irrational. □

Theorem 6.4.3 (Apéry, 1979). *The number*

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots = 1.$$

is irrational.

6.5 Transcendence

Theorem 6.5.1 (Hermite, 1873). *Euler's number $e = 2.71828\ldots$ is transcendental.*

Proof. We suppose that e satisfies a polynomial equation

$$a_d e^d + \cdots + a_1 e + a_0 = 0,\ d \in \mathbb{N},\ a_i \in \mathbb{Z},\ a_0 \neq 0\,,$$

and deduce a contradiction. (We can achieve nonzero constant term in the equation by dividing out a power of e.) We use the property of e that $(e^x)' = e^x$. Using it and integration by parts one proves by induction on $n \in \mathbb{N}_0$ that $\int_0^{+\infty} x^n e^{-x}\,dx = n!$. Hence, more generally, for every polynomial $p(x) = b_0 + b_1 x + \cdots + b_n x^n \in \mathbb{Z}[x]$,

$$\int_0^{+\infty} p(x) e^{-x}\,dx = \sum_{k=0}^{n} b_k k!\,.$$

Let $p_n(x) = x^n((x-1)(x-2)\ldots(x-d))^{n+1}$, then

$$a_0 \int_0^{+\infty} p_n(x) e^{-x}\,dx = a_0(\pm d!)^{n+1} n! + r(n)\cdot(n+1)!,\ r(n) \in \mathbb{Z}\,.$$

For $k = 1, 2, \ldots, d$ we have, by the substitution $y = x - k$,

$$\begin{aligned} a_k e^k \int_0^{+\infty} p_n(x) e^{-x}\,dx &= a_k \int_{-k}^{+\infty} p_n(y+k) e^{-y}\,dy \\ &= a_k \int_{-k}^{0} \ldots\,dy + a_k \int_0^{+\infty} p_n(y+k) e^{-y}\,dy\,. \end{aligned}$$