Some properties of holonomic sequences

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Ostravice, June 12, 2008

Overview:

1. definition and examples of holonomic sequences
2. closure properties
3. two nice holonomicity theorems in enumeration
4. non-holonomic sequences

Parts 1–3 is a review; part 4 is a joint work with Jason Bell (SFU, Vancouver), Stefan Gerhold (TU, Vienna) and Florian Luca (UNAM, Morelia).

appeared in: JB, SG, MK and FL, Non-holonomicity of sequences defined via elementary functions, *Annals of Combinatorics* **12** (2008) 1–16.

(help yourself to copies)

A sequence $(a_n) = (a_1, a_2, \ldots) \subset \mathbb{C}$ is *holonomic* (other terms are P-recursive, D-finite) if there are complex polynomials $c_0, c_1, \ldots, c_k$, $c_k \not\equiv 0$, such that

$$c_k(n)a_{n+k} + c_{k-1}(n)a_{n+k-1} + \cdots + c_0(n)a_n = 0$$

holds for every n.

Examples. 1. Fibonacci numbers: $a_{n+2} - a_{n+1} - a_n = 0$.

2. Catalan numbers: $(n+2)a_{n+1} - (4n+2)a_n = 0$, $a_n = \frac{1}{n+1}\binom{2n}{n}$.

3. Factorials: $a_{n+1} - (n+1)a_n = 0$, $a_n = n!$

4. Numbers a_n of permutations π of $1, 2, \ldots, n$ with no increasing subsequence of length 5: $(n+4)(n+3)^2 a_n - (20n^3 + 62n^2 + 22n - 24)a_{n-1} + 64n(n-1)^2 a_{n-2} = 0$.

Holonomic sequences appear in enumerative combinatorics and number theory. We are interested in asymptotic and arithmetic properties.

Closure properties. **1.** For a sequence (a_n) consider the generating power series $A = \sum a_n x^n$. Then

(a_n) is holonomic $\iff$ A satisfies a LDE with polyn. coeffs

$$g_d A^{(d)} + g_{d-1} A^{(d-1)} + \cdots + g_0 A = 0, \; g_d \not\equiv 0, \; g_i \in \mathbb{C}[x].$$

2. Thus the set of holonomic power series forms an algebra over $\mathbb{C}(x)$. This means that if (a_n) and (b_n) are holonomic then so are $(\alpha a_n + \beta b_n)$ and $(a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n)$ (Cauchy product).

3. If (a_n) and (b_n) are holonomic then so is $(a_n b_n)$ (Hadamard product).

4. If $A = \sum a_n x^n$ is algebraic (i.e., $P(x, A) = 0$ for some nonzero polynomial $P(x, y)$) then (a_n) is holonomic.

5. If $A = \sum a_n x^n$ is holonomic and $B = \sum b_n x^n$, $b_0 = 0$, is algebraic then $A(B)$ is holonomic.

6. In general composition and division of power series do not preserve holonomicity.

Holonomicity in several variables. A power series $A = \sum a_{m,n} x^m y^n$ is holonomic if the vector space of its partial derivatives

$$\{\partial^{k+l} A / \partial x^k \partial y^l \mid k, l \geq 0\}$$

has a finite dimension over $\mathbb{C}(x, y)$. Similarly for more variables.

7. (L. Lipshitz, 1988) If $A = \sum a_{m,n} x^m y^n$ is holonomic then so is the diagonal $\sum a_{n,n} x^n$. Same for more variables.

Two holonomicity results in enumeration. Both are due to I. Gessel in 1990.

1. Permutations without long increasing subsequences. Let $a_{n,k}$ be the number of permutations $\pi = p_1 p_2 \dots p_n$ of $1, 2, \dots, n$ such that $p_{i_1} < p_{i_2} < \dots < p_{i_k}$ does not hold for any k-tuple $1 \le i_1 < i_2 < \dots < i_k \le n$. Then for any fixed k, the sequence $(a_{n,k})$ is holonomic.

2. Graphs with restricted degrees. A degree $\deg_G(v)$ of a vertex $v \in V$ in a graph $G = (V, E)$ (here $E \subset \binom{V}{2} = \{e \subset V \mid |e| = 2\}$) is the number of edges in E incident with v. For $P \subset \mathbb{N}$, let $a_{n,P}$ be the number of graphs G with the vertex set $\{1, 2, \dots, n\}$ and such that $\deg_G(v) \in P$ for every $v \in \{1, 2, \dots, n\}$. Then for any fixed finite P, the sequence $(a_{n,P})$ is holonomic.

Non-holonomic sequences. How to show that (a_n) is not holonomic? Let $A = \sum a_n x^n$.

Too fast growth. (a_n) holonomic $\Rightarrow$ $|a_n| < c(n!)^d$. Thus, for example, the numbers of all graphs on $\{1, 2, \ldots, n\}$, $2^{n(n-1)/2}$, are non-holonomic.

Too many singularities. $A = \sum a_n x^n$ holonomic $\Rightarrow$ $A(x)$ has only finitely many singularities. Thus, for example,

$$\frac{x}{e^x - 1} = \sum \frac{B_n}{n!} x^n \quad \text{and} \quad \prod \frac{1}{1 - x^k} = \sum p_n x^n,$$

the Bernoulli numbers (B_n) and the partition numbers (p_n), are non-holonomic.

Too lacunary. (a_n) holonomic and $a_n = a_{n+1} = \cdots = a_{n+m} = 0$ for arbitrarily large m $\Rightarrow$ $(a_n) \equiv 0$. Thus, for example, $\sum x^{n^2}$ is non-holonomic.

Sequences given by values of elementary functions. What about sequences like $(a_n) = (\sqrt{n})$ or $(a_n) = (\log n)$?

The sequence has strange asymptotics. This method is due to P. Flajolet, S. Gerhold and B. Salvy (*Electr. J. Combin.*, 2005) and has two ingredients.

1. Abelian theorems: $a_n \sim c(n)$ as $n \to \infty \rightsquigarrow \sum a_n x^n \sim C(x)$ as $x \to \alpha$ where α is a singularity.

2. Structure theorem for solutions of DE: Every solution of a LDE with polyn. coeffs has a restricted asymptotic expansion near singularity.

Thus if the asymptotics of a_n, $n \to \infty$, is known and by 1 gives an asymptotic expansion of $\sum a_n x^n$ near singularity not of the form described in 2, then (a_n) is non-holonomic.

Examples. The sequences (p_n), p_n being the n-th prime, $(\log n)$ and (n^α), $\alpha \in \mathbb{C}\backslash\mathbb{Z}$, are non-holonomic.

A simpler proof that $(\log n)$ **is non-holonomic.** (MK, 2005). Suppose that $(\log n)$ is holonomic. Thus, for some polynomials $a_0, a_1, \ldots, a_k$ where $a_0 a_k \not\equiv 0$, the function

$$F(x) := a_0(x)\log(x) + a_1(x)\log(x+1) + \cdots + a_k(x)\log(x+k)$$

has infinitely many zeros at $x = 1, 2, \ldots$. Note that $F \not\equiv 0$ because $F \equiv 0$ would imply that $\log x$ is meromorphic at $x = 0$ which is not. By Rolle's theorem, all derivatives $F', F'', \ldots$ have infinitely many real zeros as well. But, denoting $d = \max \deg a_i$, it follows that $F^{(d+1)}$ is a rational function (we killed all logs). Since it has infinitely many zeros, $F^{(d+1)} \equiv 0$ and F is a polynomial with degree $\leq d$. But this is absurd (again, $\log x$ is not meromorphic at $x = 0$). $\square$

Elementary functions have finitely many zeros. This is the method of our paper. Suppose that $f(x)$ is a nice function. To prove that $(f(n))$ is non-holonomic, we proceed in two steps.

Step 1. We show that if f is not $\equiv 0$ and not all polynomials $a_0, a_1, \ldots, a_k$ are $\equiv 0$, then

$$F(x) := a_0(x)f(x) + a_1(x)f(x+1) + \cdots + a_k(x)f(x+k)$$

is $\not\equiv 0$.
Step 2. We show that if $F(x)$ vanishes at all $x \in \mathbb{N}$ then $F \equiv 0$.

In step 1 we use meromorphicity, singularities, growth conditions. In step 2 we use a result of A. Khovanskii (book *Fewnomials*, 1991): Every elementary real function not involving in its definition $\sin x$ and $\cos x$ (or involving them so that their domains of definition are bounded) has only finitely many simple zeros in its domain of definition.

For example, we get the following result.

Theorem (JB, SG, MK and FL, 2008). Let the function $f \in \mathbb{R}(x, \log x, \arctan x)$ be analytic on $(0, +\infty)$. Then $(f(n))$ is holonomic $\iff f \in \mathbb{R}(x)$.

Thank you for your attention!