**Counting even and odd partitions**

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**1. Introduction.** It is a lovely fact that $[n] = \{1, 2, \ldots, n\}$, $n \geq 1$, has as many subsets $X$ with an even cardinality $|X|$ as those with an odd cardinality, namely $2^{n-1}$ of both. To prove it, pair every subset $X$ with $X \pm 1$ where $X \pm 1$ is $X \backslash \{1\}$ if $1 \in X$ and $X \cup \{1\}$ if $1 \not\in X$. Then $X \mapsto X \pm 1$ is an involution that changes the parity of $|X|$ and the result follows.

More generally, in enumerative combinatorics one often has a family $\mathcal{S}_n$ of objects on $[n]$ such that every object $X$ has a natural *size* $s(X) \in \mathbf{N}_0$ of some kind. Then besides the total number of objects $S_n = |\mathcal{S}_n|$ one can consider also

$$S_n^{\pm} = \sum_{X \in \mathcal{S}_n} (-1)^{s(X)},$$

the surplus of the objects with an even size over those with an odd size. For subsets of $[n]$ and $s(X) = |X|$ we have $S_n^{\pm} = 0$ for every $n \geq 1$ (but $S_0^{\pm} = 1$). In this note we present to the reader four examples of the described situation. We investigate the corresponding numbers $S_n^{\pm}$ by means of generating functions, an analytic continuation argument, and, again, the involution trick. Our first example is a classics but the other three are much less known.

**2. Integer partitions.** $\mathcal{S}_n$ consists of the partitions $X$ of $n$ into distinct parts, $n = a_1 + a_2 + \cdots + a_k$ where $a_1 > a_2 > \ldots > a_k \geq 1$ are integers, and $s(X) = k$ is just the number of parts.

**Theorem 1.** (L. Euler, 1748) For integer partitions with distinct parts, $S_n^{\pm} = (-1)^m$ if $n = \frac{1}{2}m(3m \pm 1)$ and $S_n^{\pm} = 0$ else.

This is Euler's celebrated pentagonal identity which can be written equivalently as

$$\prod_{n=1}^{\infty}(1 - x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{m(3m+1)/2}.$$

Franklin's famous 1881 proof using the involution trick is reproduced in the book [1] of Andrews or in Hardy and Wright [4].

**3. Noncrossing set partitions.** A (set) partition of $[n]$ is a collection $X = \{B_1, B_2, \ldots, B_k\}$ of nonempty disjoint subsets of $[n]$, called *blocks*, whose union is $[n]$. It is *crossing* if there are four numbers $1 \le a < b < c < d \le n$ and two distinct blocks $A, B \in X$ such that $a, c \in A$ and $b, d \in B$. Else $X$ is a *noncrossing* partition. $\mathcal{S}_n$ consists of the noncrossing partitions of $[n]$ and $s(X) = k$ is just the number of blocks. Kreweras [5] proved that $S_n = |\mathcal{S}_n| = \frac{1}{n+1}\binom{2n}{n}$, the $n$th Catalan number. The survey [10] of Simion contains much information on the combinatorics of noncrossing partitions.

**Theorem 2.** For noncrossing set partitions, $S_n^{\pm} = (-1)^{m+1}\frac{1}{m+1}\binom{2m}{m}$ if $n = 2m+1$ and $S_n^{\pm} = 0$ if $n = 2m$.

*Proof.* Let

$$F = F(x, y) = \sum_{n \ge 0} \sum_{X \in \mathcal{S}_n} x^n y^{s(X)} = 1 + xy + x^2(y + y^2) + \cdots.$$

We are interested in

$$G = G(x) = \sum_{n \ge 0} S_n^{\pm} x^n.$$

Clearly, $G(x) = F(x, -1)$. We show that

$$F = 1 + xyF + xF(F - 1). \tag{1}$$

The empty $X$ is represented by 1. Now let $X$ be a noncrossing partition of $[n]$, $n \ge 1$, and $A$, $1 \in A$, be its first block. Either $|A| = 1$ or $|A| > 1$. In the former case, $A = \{1\}$ and after peeling off $A$ we obtain a noncrossing partition whose length and size is by 1 smaller. This is captured by the term $xyF$. In the latter case, we let $a$ denote the second element of $A$ and decompose $X$ into two partitions $X_1$ and $X_2$, where $X_1$ is induced by $X$ on the interval $[2, a-1]$ and $X_2$ is induced on $[a, n]$. Both $X_i$ are noncrossing. $X_1$ may be empty but $X_2$ is nonempty. Since no block intersects both intervals, $s(X_1) + s(X_2) = s(X)$. This decomposition is captured by the last term $xF(F - 1)$.

Setting in (1) $y = -1$ and rearranging, we get the equation $xG^2 - (1 + 2x)G + 1 = 0$. Thus ($G(0) = 1$)

$$G(x) = 1 + \frac{1}{2x}\left(1 - \sqrt{1 + 4x^2}\right).$$

Binomial expansion yields the stated formula for $S_n^{\pm}$. Note that setting in (1) $y = 1$, we recover the result of Kreweras. □

Is there a proof using involutions?

**4. All set partitions.** Now $\mathcal{S}_n$ consists of all partitions of $[n]$ and $s(X) = k$ is again the number of blocks. The total numbers $S_n$ are the *Bell numbers*

$$1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, \ldots$$

forming sequence A000110 of [11]. They grow superexponentially, $\log S_n = n(\log n - \log\log n + O(1))$. See de Bruijn [2, p. 108] or Lovász [6, Problem 1.9b] for more precise asymptotics. We show that $S_n^{\pm}$ remain superexponential.

**Theorem 3.** For all set partitions, if $c > 0$ is any constant then $|S_n^{\pm}| > c^n$ for some (in fact, infinitely many) $n \in \mathbf{N}$.

*Proof.* We begin with the classical expansion (see, for example, Stanley [12, p. 34])

$$G_k(x) = \sum_{n\geq 0} S(n,k)x^n = \frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}$$

where $S(n,k)$, the Stirling number of the second kind, is in our language simply the number of $X \in \mathcal{S}_n$ with $s(X) = k$ blocks. Thus

$$F(x) = \sum_{n\geq 0} S_n^{\pm} x^n = \sum_{k\geq 0} (-1)^k G_k(x) = \sum_{k\geq 0} \frac{(-x)^k}{(1-x)(1-2x)\ldots(1-kx)}.$$

Considering the action of the substitution $x := x/(1-x)$ on this expansion, we obtain the equation

$$F(x) = 1 - \frac{x}{1-x} F(x/(1-x)). \tag{2}$$

Substituting now $x := x/(1+x)$ and solving the resulting equation for $F(x)$, we obtain the second equation

$$F(x) = \frac{1}{x}\Big(1 - F(x/(1+x))\Big). \tag{3}$$

If $|S_n^{\pm}| < c^n$ for all $n \in \mathbf{N}$ for a constant $c > 0$, the series $F(x)$ has radius of convergence $r \geq 1/c > 0$ and defines in the disc $|z| < r$ an analytic function $F(z)$. However, we show that $r > 0$ is contradicted by the equations (2) and (3). Thus $|S_n^{\pm}| < c^n$ holds for no $c > 0$ and our theorem follows.

Suppose, for the contradiction, that $r > 0$. We can assume that $r \le 1$ (certainly $|S_n^{\pm}| \ge 1$ infinitely often). Let $\alpha \in \mathbf{C}$, $|\alpha| = r$, be a singularity of $F(z)$ on the circle of convergence. If $|\alpha/(1-\alpha)| < r$, we use (2) to continue $F(z)$ analytically to a neighborhood of $\alpha$, which contradicts the definition of $\alpha$. Clearly, $|\alpha/(1-\alpha)| < r$ is equivalent to $\mathrm{Re}(\alpha) < r^2/2$ and therefore for $\mathrm{Re}(\alpha) < r^2/2$ we have a contradiction. Similarly, if $|\alpha/(1+\alpha)| < r$, which is equivalent to $\mathrm{Re}(\alpha) > -r^2/2$, we use (3) to obtain the same contradiction. (Since $\alpha \ne 1$ in the former case, $\alpha \ne -1$ in the latter case, and always $\alpha \ne 0$, the bad arguments $z = -1, 0, 1$ do not bother us.) For every location of $\alpha$ (2) or (3) leads to a contradiction. (In the strip $|\mathrm{Re}(z)| < r^2/2$ one can use both equations.) Hence $r = 0$. $\square$

The numbers $S_n^{\pm}$, $n \ge 1$,

$$-1, 0, 1, 1, -2, -9, -9, 50, 267, 413, -2180, -17731, -50533, 110176, \ldots$$

form sequence A000587 of [11]. Recently their asymptotics was investigated by Yang [15] (see [11] for more references on them) who mentions that Subbarao and Verma proved that in fact $\limsup \log |S_n^{\pm}|/(n \log n) = 1$. Is $S_n^{\pm}$ zero infinitely often? This question is in [15] atributed to H. S. Wilf. Is $S_n^{\pm}$ ever zero besides $n = 2$?

**5. Matchings and crossings.** Perhaps the lack of cancelation was caused by the rapid growth of $S_n$? Our last example shows that $S_n^{\pm}$ can be small even if $S_n$ are superexponential. Now $\mathcal{S}_n$ consists of all partitions $X$ of $[2n]$ into $n$ two-element blocks. We call such $X$ *matchings* and their blocks *edges*. The size $s(X)$ is the number of crossing pairs $A, B$ of the edges of $X$ (we have defined crossing in the second example). It is easy to see that $S_n = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$: $S_n = (2n-1)S_{n-1}$ because one has $2n-1$ ways to place the end of the new first edge in the spaces of an $X \in \mathcal{S}_{n-1}$. So $\log S_n = n(\log n + O(1))$. But $S_n^{\pm}$ are very small.

**Theorem 4.** For matchings whose size is measured by the number of crossings, $S_n^{\pm} = 1$ for every $n \in \mathbf{N}$.

*Proof.* For a matching $X \in \mathcal{S}_n$ the *crucial pair* is the pair of edges $A, B \in X$ such that $\min A + 1 = \min B$ and $\min A$ is as small as possible. Notice that the crucial pair is unique and that every $X$ has it except $X^* = \{\{1,2\}, \{3,4\}, \ldots, \{2n-1, 2n\}\}$. Switching $\min A$ and $\min B$ in $X$ produces

Figure 1: The involution $\Phi$.

the maching $X'$ — see Figure 1. It is clear that $A, B$ remains the crucial pair of $X'$ and that $s(X) - s(X') = \pm 1$ because the set of crossing pairs of $X$ and that of $X'$ differ exactly in the pair $A, B$. So $\Phi : X \mapsto X'$ is an involution that changes the parity of $s(X)$. It pairs even and odd matchings except $X^*$ and $s(X^*) = 0$ is even. $\Box$

A remarkable formula for the generating polynomial counting matchings by crossings was derived by Touchard and Riordan [14, 9] and was later proved bijectively by Penaud [8]:

$$\sum_{X \in \mathcal{S}_n} x^{s(X)} = \frac{1}{(1-x)^n} \sum_{k=-n}^{n} (-1)^k \binom{2n}{n-k} x^{k(k-1)/2}.$$

The reader is invited for an exercise: recover the above formulas for $S_n$ and $S_n^{\pm}$ from the polynomial by setting $x = 1$ and $x = -1$.

**6. Concluding remarks.** Theorem 2 follows from the equation (1) that is proved in [10, p. 373]. Our derivation is more condensed. The analytic argument proving Theorem 3 seems new. So is perhaps the involution proof of Theorem 4 but the result itself, that $S_n^{\pm} = 1$, was found already by Riordan [9, p. 219]. We conclude by a problem on *connected* matchings. These are matchings $X$ with this property: For every two distinct edges $A, B \in X$ there is a chain of edges $A_0, A_1, \ldots, A_k$ of $X$ such that $A_0 = A$, $A_k = B$, and $A_i, A_{i+1}$ is a crossing pair for every $i = 0, 1, \ldots, k-1$. So both $X$ and $X'$ in Figure 1 are disconnected, having two and three components, respectively. Let $\mathcal{S}_n$ be the set of all connected matchings on $[2n]$ and $s(X)$ be again the number of crossings. It is known and not too difficult to prove, see the articles of Stein [13] and Nijenhuis and Wilf [7], that the numbers $(S_n)_{n \geq 1} = (1, 1, 4, 7, 248, 2830, \ldots)$ (A000699 of [11]) follow the recurrence $S_n = (n-1)\sum_{i=1}^{n-1} S_i S_{n-i}$. (For further results on matchings and crossings see Flajolet and Noy [3].) Now, as for $S_n^{\pm}$, do we have nice cancelation in the style of Theorems 1, 2, and 4 or do we have rather erratic behaviour as in Theorem 3?

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