Entire function is globally analytic

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Abstract

We give a minimalistic proof of the theorem in the title.

The Theorem

We prove from first principles the fundamental property of holomorphic functions. Then we add a few comments.

Theorem. *For every function $f:\ \mathbb{C} \to \mathbb{C}$ such that $f'(z) \in \mathbb{C}$ exists for every $z \in \mathbb{C}$ there are complex numbers $(a_0, a_1, \dots) \subset \mathbb{C}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

holds for every $z \in \mathbb{C}$.

By the assumption, for any $z \in \mathbb{C}$ there is a value $f'(z) \in \mathbb{C}$ such that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\frac{f(z)-f(u)}{z-u} - f'(z)| < \varepsilon$ whenever $0 < |z-u| < \delta$. The conclusion says that for every $z \in \mathbb{C}$, $a_0 + a_1 z + \cdots + a_n z^n \to f(z)$ as $n \to \infty$. A function $f:\ U \to \mathbb{C}$ on an open set U is *holomorphic (on U)* if $f'(z)$ exists for every $z \in U$. U will denote a nonempty open subset of $\mathbb{C}$. An *entire function* is holomorphic on $\mathbb{C}$. Theorem asserts that every entire function is a sum of power series.

The proof

Let $a, b \in \mathbb{C}$, $a \neq b$, be a pair of points and $f:\ U \to \mathbb{C}$ be a continuous function such that U contains the segment $S = S_{a,b} = \{\varphi(t) = a + t(b-a) \mid t \in [0,1]\}$ spanned by a and b. A *partition P of S* is a tuple of points $P = (a_0, a_1, \dots, a_k)$ on S with $a_i = \varphi(t_i)$ for some $0 = t_0 < t_1 < \cdots < t_k = 1$. An *equipartition* P has $t_i = i/k$, $i = 0, 1, \dots, k$, so $a_i - a_{i-1} = (b-a)/k$. We set $\|P\| = \max_{1 \le i \le k} |a_i - a_{i-1}|$. *Cauchy's sum (corresponding to P and f)* is

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$R(P,f) = \sum_{i=1}^{k} f(a_i)(a_i - a_{i-1})$. If $P_1, P_2, \ldots$ is a sequence of partitions of S such that $\|P_n\| \to 0$, we define the *integral of f over S* as

$$I_{a,b}(f) = \lim_{n\to\infty} R(P_n, f) \ .$$

Below we prove that the limit exists and does not depend on the choice of P_n; for P_n one may take equipartitions with $k = n$.

By a *rectangle* $R \subset \mathbb{C}$ we mean an axis-parallel rectangle with nonzero height and width. We take its vertices as a quadruple $a, b, c, d \in \mathbb{C}$, ordered counter-clockwise starting from the lower left corner a. If $f : U \to \mathbb{C}$ is continuous and the boundary ∂R of R lies in U, we define the *integral of f over ∂R* as

$$I_R(f) = I_{a,b}(f) + I_{b,c}(f) + I_{c,d}(f) + I_{d,a}(f) \ .$$

We begin by deriving some properties of $I_{a,b}(f)$ and $I_R(f)$.

Proposition 1. *Let $f, g : U \to \mathbb{C}$ be continuous functions, $S = S_{a,b} \subset U$ be a segment, and R be a rectangle with $\partial R \subset U$. The limit defining $I_{a,b}(f)$ always exists and is independent of the choice of the partitions P_n. Further we have the following properties.*

1. *If $\alpha, \beta \in \mathbb{C}$ then $I_{a,b}(\alpha f + \beta g) = \alpha I_{a,b}(f) + \beta I_{a,b}(g)$ and similarly for $I_R(\cdot)$.*
2. *$|I_{a,b}(f)| \le \max_{z\in S} |f(z)| \cdot |b-a|$ and $|I_R(f)| \le \max_{z\in\partial R} |f(z)| \cdot p(R)$ where $p(R)$ is the perimeter of R.*
3. *One has $I_{a,b}(f) = -I_{b,a}(f)$ and if $c \in S$, $c \ne a, b$, then $I_{a,b}(f) = I_{a,c}(f) + I_{c,b}(f)$.*

Proof. It suffices to show that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if P_1 and P_2 are partitions of S with $\|P_1\|, \|P_2\| < \delta$ then $|R(P_1, f) - R(P_2, f)| < \varepsilon$. Let first $P_1 \subset P_2$, $P_1 = (a_0, a_1, \ldots, a_k)$ and $\|P_1\| < \delta$. Then there are partitions Q_i of $S_i = S_{a_{i-1},a_i}$, $i = 1, 2, \ldots, k$, (using points in P_2) such that $R(P_2, f) = \sum_{i=1}^{k} R(Q_i, f)$. Denoting by h the function that is constantly $f(a_i)$ on S_i we have $m_i := |f(a_i)(a_i - a_{i-1}) - R(Q_i, f)| = |R(Q_i, h - f)| \le \max_{z\in S_i} |f(a_i) - f(z)| \cdot |a_i - a_{i-1}|$. Since continuity on a compact set implies uniform continuity, for small δ we have $|f(a_i) - f(z)| < \varepsilon/|b-a|$ for every $z \in S_i$ and $i = 1, 2, \ldots, k$. Thus $m_i < \varepsilon\frac{|a_i - a_{i-1}|}{|b-a|}$ and $|R(P_1, f) - R(P_2, f)| \le m_1 + \cdots + m_k < \varepsilon$. If P_1 and P_2 are incomparable by inclusion, we take the partition $P_3 = P_1 \cup P_2$ and apply the previous result on the pairs P_1, P_3 and P_2, P_3.

1. This follows from the linearity of $R(P, f)$ in f.

2. This follows from the bound $|R(P, f)| \le \max_{z\in S} |f(z)| \cdot |b - a|$ (we used it already), which follows from $\sum_i |a_i - a_{i-1}| = |a_k - a_0| = |b - a|$ by the collinearity of a_i.

3. This follows from the corresponding identities for Cauchy's sums, which are immediate. □

Proposition 2 (Cauchy's theorem). *Let $f : U \to \mathbb{C}$ be a continuous function and $R \subset \mathbb{C}$ be a rectangle.*

1. *If $\partial R \subset U$ and f is linear then $I_R(f) = 0$.*

2. *If $R \subset U$ (the whole R lies in U, not just the boundary) and f is holomorphic then $I_R(f) = 0$.*

Proof. For brevity we often omit in this proof the argument (f) in $I_R(f)$ and similar expressions.

1. Let $f(z) = \alpha z + \beta$ and the vertices of R be a, b, c, d. If P is an equipartition of $S_{a,b}$ then $R(P, f) = \sum_{i=1}^{k}(\alpha a_i + \beta)(a_i - a_{i-1}) = \alpha \sum_{i=1}^{k}(a + i(b-a)/k)(b-a)/k + \beta(b-a) = \alpha(a(b-a) + (\frac{1}{2} + \frac{1}{2k})(b-a)^2) + \beta(b-a)$. Hence $k \to \infty$ gives $I_{a,b} = \alpha(b^2 - a^2)/2 + \beta(b-a) = g(b) - g(a)$ where $g(z) = \alpha z^2/2 + \beta z$. So $I_R = I_{a,b} + I_{b,c} + I_{c,d} + I_{d,a} = (g(b) - g(a)) + (g(c) - g(b)) + (g(d) - g(c)) + (g(a) - g(d)) = 0$.

2. We divide R in four congruent rectangles $R_1, \ldots, R_4$ (by joining the midpoints of the opposite sides of R) and conclude that for some $j \in \{1, 2, 3, 4\}$, $|I_{R_j}| \geq |I_R|/4$. This follows from the triangle inequality and the identity $I_R = I_{R_1} + \cdots + I_{R_4}$ (which follows from the definition of I_{R_i} by using part 3 of Proposition 1). Clearly, $p(R_j) = p(R)/2$. Iterating this division we obtain nested rectangles $R = R_0 \supset R_1 \supset \ldots$ such that $|I_{R_n}| \geq |I_R|/4^n$ and $p(R_n) = p(R)/2^n$. We take the point $z_0 \in U$ given by

$$\{z_0\} = \bigcap_{n=0}^{\infty} R_n \; .$$

Since $f'(z_0)$ exists, for given $\varepsilon > 0$ for large enough n and $z \in R_n$ one has $f(z) = f(z_0) + f'(z_0)(z - z_0) + \Delta(z)(z - z_0)$ with $|\Delta(z)| < \varepsilon$. Let $g(z) = f(z_0) + f'(z_0)(z - z_0)$ and $h(z) = \Delta(z)(z - z_0)$. Then $f = g + h$ and (by part 1 of Proposition 1) $I_{R_n}(f) = I_{R_n}(g) + I_{R_n}(h) = I_{R_n}(h)$ because $I_{R_n}(g) = 0$ by part 1 of this proposition. Now $|h(z)| < \varepsilon p(R_n)$ for $z \in R_n$. Thus (by part 2 of Proposition 1)

$$|I_R|/4^n \leq |I_{R_n}| = |I_{R_n}(h)| < \varepsilon p(R_n)^2 = \varepsilon p(R)^2/4^n$$

and $|I_R| < \varepsilon p(R)^2$. So $I_R(f) = 0$. □

Proposition 3. *Let $R \subset \mathbb{C}$ be the square with vertices $\pm 1 \pm i$. Then $\rho := I_R(1/z) \neq 0$.*

Proof. Now $a = -1 - i$, $b = 1 - i$, $c = 1 + i$, and $d = -1 + i$. Let P be an equipartition of $S_{a,b}$. Then

$$R(P, 1/z) = \sum_{j=1}^{k} \frac{2/k}{a + 2j/k} = \sum_{j=1}^{k} \frac{(2/k)(2j/k - 1 + i)}{(2j/k - 1)^2 + 1}$$

and we see that $\mathrm{Im}(R(P, 1/z)) \geq 1$. Thus $\mathrm{Im}(I_{a,b}(1/z)) \geq 1$ and in particular $I_{a,b}(1/z) \neq 0$. Now comes perhaps the crucial point of the whole proof. Since $\frac{2/k}{a+2j/k} = \frac{2i/k}{ai+2ij/k} = \frac{2i/k}{b+2ij/k}$ and $c - b = 2i$, we have $R(P, 1/z) = R(iP, 1/z)$

where iP is the partition of $S_{b,c}$ obtained by rotating P around 0 by i. Thus $I_{b,c}(1/z) = I_{a,b}(1/z)$, and similar arguments (extending the fraction by -1 and $-i$) show that also $I_{c,d}(1/z) = I_{a,b}(1/z)$ and $I_{d,a}(1/z) = I_{a,b}(1/z)$. Hence $I_R(1/z) = 4I_{a,b}(1/z) \neq 0$ (and $\mathrm{Im}(I_R(1/z)) \geq 4$). $\Box$

For $a \in \mathbb{C}$ we set $\mathbb{C}_a = \mathbb{C}\backslash\{a\}$ and denote by H_a the set of all holomorphic functions $f:\ \mathbb{C}_a \to \mathbb{C}$. Let $H = \bigcup_{a\in\mathbb{C}} H_a$. We define the functional

$$I:\ H \to \mathbb{C},\ I(f) = I_R(f),\ f \in H_a,\ a \in \mathrm{int}(R)$$

— $R \subset \mathbb{C}$ is any rectangle containing a in its interior.

Proposition 4. *The mapping $I:\ H \to \mathbb{C}$ is correctly defined, its value does not depend on the choice of R, and has the following properties.*

1. *If $\alpha, \beta \in \mathbb{C}$ and $f, g \in H_a$ then $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$.*
2. *If $f \in H_a$ and is bounded near a then $I(f) = 0$.*
3. *For $(z-a)^{-1} \in H_a$ one has $I((z-a)^{-1}) = \rho \neq 0$ (ρ is defined in Proposition 3).*
4. *Suppose that $r > 0$ is a constant and $f \in H_a$ and $f_n \in H_b$, $n = 1, 2, \ldots$, are such that $f_n(z) \to f(z)$ uniformly in $z \in A$ for every bounded set A lying in $|z| > r$. Then $I(f_n) \to I(f)$.*

Proof. Let $f \in H_a$ and $R, S \subset \mathbb{C}$ be two rectangles with a in their interiors. Assume first that S lies in the interior of R. Using lines extending the sides of S we divide R in nine rectangles $R_1, \ldots, R_9$ with S being one of them, say $S = R_5$. As in the proof of part 2 of Proposition 2 we have $I_R(f) = I_{R_1}(f) + \cdots + I_{R_9}(f)$. For $i \neq 5$ we have $I_{R_i}(f) = 0$ by part 2 of Proposition 2 because $R_i \subset \mathbb{C}_a$. So $I_R(f) = I_{R_5}(f) = I_S(f)$. The general position of R and S reduces to two applications of the just discussed situation simply by sufficiently shrinking one of the rectangles. So $I_R(f) = I_S(f)$ whenever $a \in \mathrm{int}(R) \cap \mathrm{int}(S)$.

1. This follows from part 1 of Proposition 1.

2. Immediate from the bound in part 2 of Proposition 1 and the independence of I on R.

3. This follows from Proposition 3 by the shift $z \mapsto z + a$.

4. Let R be any rectangle containing both a and b in its interior and with boundary lying in $|z| > r$. Then $I(f) - I(f_n) = I_R(f - f_n) \to 0$ for $n \to \infty$ by the assumption on f_n and the modulus–perimeter bound on $|I_R(f - f_n)|$. (Note that $I(f - f_n)$ is syntactically incorrect because $f - f_n \notin H$ when $a \neq b$.) $\Box$

Proposition 5 (Cauchy's formula). *Let $f:\ \mathbb{C} \to \mathbb{C}$ be entire. Then for every $a \in \mathbb{C}$ we have*

$$f(a) = \frac{I(f(z)/(z-a))}{\rho}\ .$$

Proof. Let $a \in \mathbb{C}$ be fixed. By parts 1, 3 and 2 of Proposition 4,

$$I(f(z)/(z-a)) = I(f(a)/(z-a)) + I((f(z)-f(a))/(z-a)) = \rho f(a) + 0 .$$

$\square$

Proposition 6. *Let $f : \mathbb{C} \to \mathbb{C}$ be entire. Then for every $a \in \mathbb{C}$ we have*

$$f(a) = \sum_{n=0}^{\infty} \frac{I(f(z)/z^{n+1})}{\rho} a^n \text{ (ρ is defined in Proposition 3)} .$$

Proof. Let $a \in \mathbb{C}$ be fixed. By the geometric series, we have uniformly in $z \in \mathbb{C}$ with $|a/z| < 1 - \delta$, $\delta \in (0, 1)$, that

$$\frac{1}{z-a} = \frac{1}{z} \cdot \frac{1}{1-a/z} = \sum_{n=0}^{\infty} \frac{a^n}{z^{n+1}} .$$

Substituting it in Cauchy's formula and using part 1 and part 4 of Proposition 4 (with $b = 0$ and functions $f(z)/(z-a)$ and $f_n(z) = f(z) \sum_{k=0}^{n} a^k/z^{k+1}$, we also use that $f(z)$ is bounded on any bounded set) we get the stated power series expansion. $\square$

We are done! Theorem is proven, the coefficients are $a_n = I(f(z)/z^{n+1})/\rho$.

Concluding comments

The basic property of holomorphic functions stated in Theorem enters the game in Bak and Newman [1] at page 60, in Henrici [4] at page 329, in Kriz and Pultr [5] at page 248 (complex analysis is not the main topic of this book), in Rudin [7] at page 207, in Veselý [9] at page 106, and we could continue with many more monographs. But certainly one does not need 60 pages to prove it? How complicated and long would be a selfcontained proof from first principles? Our aim was to provide such a proof and now the reader may judge it and try to improve upon it. In the proof we followed (critically) [1], with a look in [5]. The proof of part 2 of Proposition 2 is taken from [1, Theorem 4.14] (where no attribution is given). The transition from Proposition 5 to Proposition 6 is classical. For formalized proofs of Cauchy's theorem and formula see Harrison [3] (Theorem is not considered). Much of path integration is completely unnecessary to prove Theorem and very little suffices, as we demonstrated. The use of Cauchy's sums to define integrals is indeed due to Cauchy, see [2, lectures 21 and 23] (we learned it in Schwabik and Šarmanová [8]). As is well known, in the real domain Theorem fails ($f(x) = 0$ for $x \le 0$ and $f(x) = x^2$ for $x \ge 0$ has derivative everywhere, $f'(x) = 0$ for $x \le 0$ and $f'(x) = 2x$ for $x \ge 0$, but is not analytic in any neighborhood of 0) nor it holds in the p-adic domain, see Robert [6]. Perhaps unusual in our approach is Proposition 3. Of course, $I(1/z) = I_R(1/z) = 2\pi i$ but to prove it explicitly one would have to make explicit the machinery of exponential function (or trigonometric functions) and Riemann integration, which is unnecessary — nonvanishing fully suffices. The nonvanishing in Proposition 3 is equally crucial for the proof as the vanishing in Cauchy's theorem (Proposition 2). Without it there would be no Cauchy's formula to be expanded in power series. Our proof of Proposition 3 by integration over the boundary of a square demonstrates the fundamental role of the algebraic rotational 4-fold symmetry of $\mathbb{C} = \mathbb{R}[i]$, stemming from the fact that i is a (primitive) 4th root of unity, $i^4 = 1$.

References

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