Counting Pattern-free set partitions I: A generalization of Stirling numbers of the second kind

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Set partitions I

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Abstract

A partition u of $[k] = \{1, 2, \ldots, k\}$ is contained in another partition v of $[l]$ if $[l]$ has a k-subset on which v induces u. We are interested in counting partitions v not containing a given partition u or a given set of partitions R. This concept is related to that of forbidden permutations. A strengthening of Stanley–Wilf conjecture is proposed.

We prove that the GF counting v is rational if (i) R is finite and the number of parts of v is fixed or if (ii) u has only singleton parts and at most one doubleton part. In fact, (ii) is an application of (i). As another application of (i) we prove that for each k the GF counting partitions with k pairs of crossing parts belongs to $\mathbf{Z}(\sqrt{1-4x})$.

1 Introduction

An n-permutation $b_1b_2\dots b_n$, a permutation of $[n]=\{1,2,\dots,n\}$, *avoids* an m-permutation $p=a_1a_2\dots a_m$ if it has no subsequence $b_{i_1}b_{i_2}\dots b_{i_m}$ such that $b_{i_r}<b_{i_s}$ iff $a_r<a_s$. The number of n-permutations avoiding p is $S_n(p)$. Similarly, $S_n(R)$ counts n-permutations avoiding each p from a set of permutations R. For R fixed and $n=1,2,\dots,$ determine $S_n(R)$. This is the problem of *forbidden permutations* that was introduced by Simion and Schmidt [22] and further investigated in, for example, [3, 4, 5, 25, 30]. (In the wqo theory, the avoidance of permutations was considered earlier in [15, 16].)

We propose a new class of similar enumerative problems based on set partitions. A partition $v=([l],\sim_v)$ given by its equivalence relation *does not contain* $u=([k],\sim_u)$, in symbols $v\not\succ u$, if there is no increasing injection $f:[k]\to[l]$ such that $i\sim_u j$ iff $f(i)\sim_v f(j)$. For u a partition, $P(u;n,l)$ is the number of partitions of $[l]$ not containing u and having n parts. For R a set of partitions, $P(R;n,l)$ is defined in an obvious way. The problem of *forbidden partitions* is, for R fixed and $n,l=1,2,\dots,$ to determine $P(R;n,l)$.

Both problems are closely related. We encode the m-permutation $p=a_1a_2\dots a_m$ by the partition u_p of $[2m]$ with parts $\{i,m+a_i\}$. Then $S_n(p)$ is the number of the partitions u_q such that q is an n-permutation and $u_q\not\succ u_p$. In particular, $S_n(p)\le P(u_p;\cdot,2n)$ where $P(u;\cdot,l)=\sum_{n\ge1}P(u;n,l)$. A conjecture due to R. Stanley and H. Wilf says that $S_n(p)=O(c^n)$ for each p. (Recently, Bóna [5] confirmed it for many permutations.) We offer a stronger conjecture: $P(u_p;\cdot,l)=O(c^l)$ for each permutation p. If true, it also holds for each u obtained from u_p by adding some singleton parts. Such a u will be called a *sufficiently restrictive partition* or, shortly, *srp*. By Example 1, srps are the only partitions u for which $P(u;\cdot,l)$ may have an exponential upper bound.

Trivially, $S_n(12)=S_n(21)=1$. By [13, 22], $S_n(p)=\frac{1}{n+1}\binom{2n}{n}$ for each 3-permutation p. It is more complicated to determine $S_n(p)$ for a 4-permutation, see [3]. Perhaps the complexity of $P(u;\cdot,l)$ for srps with m doubletons is similar to that of $S_n(p)$ for $(m+1)$-permutations. To support the intuition, in Section 4 we prove that for each srp u with one doubleton the GF (generating function) $\sum_{l\ge1}P(u;\cdot,l)y^l$ is rational. Also, the GF for each of the two srps with two doubletons and no singletons satisfies a quadratic equation, see Examples 2 and 3.

We discuss the following topics. Section 2 introduces sequential representation of partitions. In Section 3 we prove Theorem 3.1 saying that for each n and finite R the GF

$\sum_{l\ge 1} P(R;n,l)y^l$ is a rational function of a particular kind. The induction scheme used forces us to prove a more general Theorem 3.2. In the beginning of the proof its outline is given. Theorem 3.1 is used to prove Theorem 4.1 saying that each srp with one doubleton has a rational GF. It is not a surprising result but it may be of some interest as a first step in measuring the complexity of $P(u;\cdot,l)$; the proofs in Section 4 are only sketched. In Section 5 we apply Theorem 3.2 to prove that the GF of partitions having a fixed number of pairs of crossing parts belongs to $\mathbf{Z}(x,\sqrt{1-4x}) = \mathbf{Z}(\sqrt{1-4x})$; this complements [6]. In Section 6 we give additional comments and pose some problems.

Forbidden partitions might shed a new light on forbidden permutations. For partitions there goes in paralell a strong branch of extremal results (see Example 5). It might be of use to crossbreed the enumerative and extremal branches.

2 Notation and examples

A partition $u = ([k], \sim_u)$ can be represented by a finite sequence $a_1a_2\dots a_k \in S^*$ over an infinite alphabet S, where S contains $\mathbf{N} = \{1,2,\dots\}$ and some letters $a,b,c,\dots$, by choosing the sequence so that $i \sim_u j$ iff $a_i = a_j$. A mapping $f : S \to S$ acts on S^* in a natural way, $f(a_1a_2\dots a_k) = f(a_1)f(a_2)\dots f(a_k)$. If $u,v \in S^*$ and $u = f(v)$ for an injection f, we say that u and v are *equivalent*. Partitions correspond to blocks of equivalent sequences. In sequel, this representation of partitions will be used.

For $u \in S^*$, $|u|$ is the length of u, $S(u) \subset S$ is the set of symbols used in u, and $\|u\|$ is the cardinality of $S(u)$ (i.e., the number of parts). Clearly, $u \prec v$ means that u is equivalent to a subsequence of v. Such a subsequence will be called a *u-copy*. Each block of equivalent sequences contains a unique *canonical* sequence, a sequence u such that (i) $S(u) = [n]$ and (ii) for each pair $1 \le i < j \le n$ the first occurrence of i in u precedes that of j. To *canonize* v means to replace it by the equivalent canonical sequence.

We remind that $P(R;n,l)$ counts canonical v such that $|v| = l$, $\|v\| = n$, and $v \not\succ u$ for each $u \in R$. The corresponding GF is denoted by

$$G(R;x,y) = \sum_{n,l\ge 1} P(R;n,l)x^ny^l.$$

For simplicity, when possible we let the parameter n unrestricted and consider only the quantities $P(R;\cdot,l)$ and $G(R;1,y)$. If $u \prec v$ then $P(u;n,l) \le P(v;n,l)$. If $\overline{u}$ is the reversal of

u then $P(\overline{u}; n, l) = P(u; n, l)$. The proofs of the formulas in the following example are easy and thus omitted.

EXAMPLE 1. With $(2j-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2j-1)$ and the convention $(-1)!! = 1$ we have

$$P(aaa; \cdot, l) = \sum_{j=0}^{\lfloor l/2 \rfloor} (2j-1)!! \binom{l}{2j}.$$

As for $u = aabb$, we have

$$P(aabb; \cdot, l) = \sum_{k \geq 0,\ p \geq 3}^{p+2k \leq l} (k+1)^2 \binom{l}{p+2k} k! \ + \ \sum_{k=0}^{\lfloor l/2 \rfloor} \binom{l}{2k} k!$$

Both $P(aaa; \cdot, l)$ and $P(aabb; \cdot, l)$ grow faster than any c^l. It is obvious already from the fact that $u_p \not\prec aaa, aabb$ for each p. The sequences aaa and $aabb$ are probably the only sequences u which have a superexponential $P(u; \cdot, l)$ and are minimal (to $\prec$) with this property.

We remind that the srps are sequences containing neither aaa nor $aabb$. By Example 1, each nonsrp u has a superexponential $P(u; \cdot, l)$. Examples of srps: 1234256 and $abcbcda$. If $u \prec v$ for a srp v, u is a srp as well. The only srps with two doubletons and no singletons are $abab$ and $abba$. Their GF's are as follows.

EXAMPLE 2. Let $u = abab$. A canonical $v, v \not\succ abab$ splits uniquely in $v = 1v_1v_2$ so that $1 \notin S(v_1)$ and v_2 starts with 1 if nonempty. Then $v_i \not\succ abab$, v_i may be empty, and $S(v_1) \cap S(v_2) = \emptyset$. On the other hand, any choice of such v_i's is admissible. Thus, $G(abab; 1, y) = y(1 + G(abab; 1, y))^2$. We obtain the classical results [2, 14]

$$G(abab; 1, y) = \frac{1 - 2y - \sqrt{1-4y}}{2y} \quad \text{and} \quad P(abab; \cdot, l) = \frac{1}{l+1}\binom{2l}{l}.$$

Partitions not containing $abab$ are now called *noncrossing partitions*. At first they were investigated by Kreweras [14] and Poupard [17]. They appear in geometric extremal problems [8, 10], poetry [2], probability theory [24], molecular biology [28, 29], enumerative bijections [7, 18], and combinatorics of the partitions lattice [14, 21, 23]; the list of references is not exhaustive. $P(abab; \cdot, l) = O(c^l)$ and the right constant is $c = 4$.

EXAMPLE 3. For $u = abba$ the GF equals, see [11],

$$G(abba; 1, y) = \frac{-y + 3y^2 - 2y^3 - y\sqrt{1 - 2y - 3y^2}}{-2 + 8y - 6y^2 + 2y^3}.$$

Again $P(abba; \cdot, l) = O(c^l)$. The right constant is $c = 1/\gamma = 3.14790\ldots$, $\gamma > 0$ being the root of $y^3 - 3y^2 + 4y - 1$ closest to the origin.

3 Fixed number of parts

Example 4. In our notation Stirling numbers of the second kind are $P(\emptyset; n, l)$. Since the canonical v's with $\|v\| = n$ arise from $12\ldots n$ by inserting a $v_1 \in \{1\}^*$ between 1 and 2, a $v_2 \in \{1, 2\}^*$ between 2 and 3, …, and a $v_n \in \{1, 2, \ldots, n\}^*$ after n, we have

$$\sum_{l \geq 1} P(\emptyset; n, l) y^l = \frac{y^n}{(1-y)(1-2y)\cdots(1-ny)}.$$

The following theorem generalizes this classical result.

Theorem 3.1 *For each $n \in \mathbf{N}$ and finite $R \subset S^*$,*

$$\sum_{l \geq 1} P(R; n, l) y^l = \frac{a(y)}{(1-y)^{r_1}(1-2y)^{r_2}\cdots(1-ty)^{r_t}},$$

where $a(y) \in \mathbf{Z}[y]$, $r_i \geq 0$, $t = \min(n, k)$, and $k = \min_{u \in R} \|u\| - 1$. For $k = 0$ the denominator is 1.

In particular, for $k = 0$ the GF is a polynomial from $\mathbf{Z}[y]$; this is obvious. For $k = 1$ the function $P(R; n, l)$ is a polynomial from $\mathbf{Q}[l]$. We look at the cases $R = \{abab\}$ and $R = \{ababa\}$ when $k = 1$.

Example 5. It is well known [14] that

$$P(abab; n, l) = \frac{1}{l-n+1}\binom{l}{n}\binom{l-1}{n-1},$$

a polynomial in l of degree $2n-2$. What changes if $R = \{ababa\}$? Sequence $w = a_1 a_2 \ldots a_l$ is called *sparse* if $a_i \neq a_{i+1}$ for each i. Sequences $v, v \not\succ ababa$ arise from a sparse $w, w \not\succ ababa$ by arbitrarily replacing terms of w by intervals of occurrences of the same symbol. Let p_j be the number of nonequivalent sparse w's, $\|w\| = n$ and $|w| = j$, not containing $ababa$, and $N_5(n) = \max\{j : p_j \neq 0\}$. (By Lemma 3.3, $N_5(n) = O(n^2)$.) Clearly, $P(ababa; n, l)$ is the coefficient at y^l in

$$\sum_{j=n}^{N_5(n)} \frac{p_j y^j}{(1-y)^j},$$

a polynomial in l of degree $N_5(n) - 1$. Unlike the analogous extremal function $N_4(n) = 2n-1$ for $abab$, the function $N_5(n)$ is difficult to handle. Here we mention only the estimate $\frac{1}{2}n\alpha(n) - 2n < N_5(n) < 2n\alpha(n) + O(n\alpha(n)^{1/2})$, where $\alpha(n)$ is the extremely slowly growing inverse of the Ackermann function. For the lower and upper bound consult [31] and [12], respectively. More information on the *Davenport–Schinzel sequences*, of which w is a particular case, can be found in [20].

Outline of the proof of Theorem 3.1. Suppose first $R = \{u\}$. We want to use induction on $|u|$. To count the v's such that $\|v\| = n$ and $v \not\succ u$, we split v in $v = v_1v_2\dots v_r$ so that the v_i's are subject to simpler constraints and can be chosen independently. A u-copy appears then in v iff u splits in $u = u_1u_2\dots u_r$ so that there is a u_i-copy of certain type in v_i. We are forced to consider a stronger induction statement involving any finite R and, for each $u \in R$, prescribed types of the u-copies in v. This is formulated in Theorem 3.2 and the preceding definitions. We work with a special R (ideal), because for induction it is better to have R closed to subsequences. Theorem 3.1 follows from Theorem 3.2 simply by summing all cases. The stronger restriction of the denominator is established in Lemmas 3.3 and 3.4. The inductive proof of Theorem 3.2 is started by Lemma 3.5, a variation on Example 4. Then we describe how the u_i-copies in $v_1v_2\dots v_r$ merge in a u-copy. Lemma 3.6 states a property of merging. Then we define the splitting $v = v_1v_2\dots v_r$ and in Lemma 3.7 state its key property. In Lemmas 3.8, 3.9, and the concluding argument we perform the induction step.

A finite $I \subset S^*$ is an *ideal* if each $u, u \prec v \in I$, is equivalent to some $w \in I$. Let $w = b_{i_1}b_{i_2}\dots b_{i_k}$ be a subsequence of $v = b_1b_2\dots b_l$ equivalent to $u = a_1a_2\dots a_k$. The *type* of the u-copy w in v is the injection $f : [\|u\|] \to [\|v\|]$ defined by canonizing u and v and then setting $f(a_j) = b_{i_j}$. So type is the injection that maps the position of every symbol in w to its position in v. Several u-copies may have the same type. All types of all u-copies in v form the set $T(u, v)$. For example,

$$T(abab, 4332421141) = \{\{(1,1),(2,3)\},\{(1,3),(2,1)\},\{(1,1),(2,4)\}\}$$

and there are six $abab$-copies in 4332421141.

Let, for $n \in \mathbf{N}$ and $R \subset S^*$, $\mathcal{F}(R, n)$ be the set of all mappings F such that F is defined on R and $F(u)$, $u \in R$, is a set of injections from $[\|u\|]$ to $[n]$.

Theorem 3.2 *Let $n \in \mathbf{N}$, I be an ideal, $F \in \mathcal{F}(I, n)$, and $P(I, F; n, l)$ count all canonical v satisfying $\|v\| = n, |v| = l$, and $T(u, v) = F(u)$ for each $u \in I$. Then*

$$\sum_{l\ge1} P(I, F; n, l)y^l = \frac{a(y)}{(1-y)^{r_1}(1-2y)^{r_2}\cdots(1-ny)^{r_n}},$$

where $a(y) \in \mathbf{Z}[y]$ and $r_i \ge 0$.

Any finite $R \subset S^*$ is easily completed to an ideal $I \supset R$. Then $P(R; n, l) = \sum_F P(I, F; n, l)$, summed over all $F \in \mathcal{F}(I, n)$ such that $F \equiv \emptyset$ on R, and Theorem 3.1

follows, with the denominator $(1-y)^{r_1}\cdots(1-ny)^{r_n}$. The same argument shows that Theorem 3.2 holds with R instead of I as well. The remaining part of Theorem 3.1, the restriction of the denominator, follows if we show that for every $u \in S^*$ and $n \in \mathbf{N}$ we have $P(u;n,l) = o(\|u\|^l)$. We prove it in the next two lemmas.

For $v \in S^*$ and $m \in \mathbf{N}$ consider the *m-splitting* $v = v_1v_2\ldots v_r$, where v_1 is the longest initial interval with $\|v_1\| \le m$, v_2 is the longest interval following v_1 with $\|v_2\| \le m$ and so on. Thus, $\|v_1\| = \cdots = \|v_{r-1}\| = m$, $\|v_r\| \le m$, and the splitting is unique.

Lemma 3.3 *If $v, \|v\| = n$ has the m-splitting with at least*

$$2(s-1)\binom{n}{m+1} + 2$$

intervals, then v contains each u satisfying $\|u\| \le m+1$ and $|u| \le s$.

Proof. Let $v = v_1v_2\ldots v_r$ be the m-splitting. We have $\|v_iv_{i+1}\| \ge m+1$ for each i and we select a subset $X_i \subset S(v_iv_{i+1}), |X_i| = m+1$. By the pigeonhole principle, $X_{2i_1-1} = X_{2i_2-1} = \cdots = X_{2i_s-1}$ for some s indices $1 \le i_1 < i_2 < \cdots < i_s \le r/2$. Taking from each $v_{2i_j-1}v_{2i_j}$ an appropriate term, we create a u-copy in v. □

Lemma 3.4 *For every $n \in \mathbf{N}$ and $u \in S^*$ we have $P(u;n,l) = O(l^{h-1}(\|u\|-1)^l)$; the constant in O and h depend only on n and u.*

Proof. Suppose that $v \not\succ u$, $\|v\| = n$, $\|u\| = m+1$, and $v = v_1v_2\ldots v_r$ is the m-splitting. By the previous lemma, $r \le h = h(u,n)$. Once the sets $S(v_i)$ are chosen, there are at most m^l possibilities for each $v_i, |v_i| = l$. To account for v_r (since $\|v_r\| \le m$) we multiply the bound by the factor m. Hence, $P(u;n,l) \le$ the coefficient at y^l in

$$m\sum_{r=1}^{h}\frac{\binom{n}{m}^r}{(1-my)^r},$$

which is $O(l^{h-1}m^l)$. □

Therefore if $u \in R$ attains the minimum $\|u\|$, the denominator cannot have a root smaller than $\frac{1}{\|u\|-1}$. This finishes the proof of Theorem 3.1.

The proof of Theorem 3.2 goes by induction on $|I|$ and starts with the ideal $I(r) = \{a, aa, aaa, \ldots, aa\ldots a\}$, the last sequence of a's having length r.

Lemma 3.5 *For each $r, n \in \mathbf{N}$ and $F \in \mathcal{F}(I(r), n)$,*

$$\sum_{l \geq 1} P(I(r), F; n, l) y^l = \frac{a(y)}{(1-y)^{r_1}(1-2y)^{r_2} \cdots (1-ny)^{r_n}},$$

where $a(y) \in \mathbf{Z}[y]$ and $r_i \geq 0$.

Proof. Let $G(\overline{x}; y)$, where $\overline{x} = (x_1, \ldots, x_n)$ with $x_i \in [r-1] \cup \{r^+, 1^+\}$, be the GF counting by length the canonical $v, \|v\| = n$, with x_i occurrences of i (1^+ means any number ≥ 1 and similarly for r^+). By the definitions, the above GF equals $G(\overline{x}; y)$ for some $\overline{x}$ with no $x_i = 1^+$. (Or it is identically 0, if the conditions imposed by F are contradictory.) Each such $G(\overline{x}; y)$ equals, by the principle of inclusion and exclusion, $\sum \pm G(\overline{x}; y)$ for some $\overline{x}$'s with no $x_i = r^+$. It suffices to show that each $G(\overline{x}; y)$ for $\overline{x}$ with no $x_i = r^+$ has the stated form.

By Example 4, the GF of canonical v's, in which y counts length and y_i counts the occurrences of i, is

$$G(y, y_1, \ldots, y_n) = \frac{y^n y_1 \cdots y_n}{(1-yy_1)(1-yy_1-yy_2)\cdots(1-yy_1-yy_2-\cdots-yy_n)}.$$

Thus, if $x_1, \ldots, x_k \in [r-1]$ and $x_{k+1} = \cdots = x_n = 1^+$, $G(\overline{x}; y)$ equals

$$\frac{\partial^{x_1+\cdots+x_k} G(y, y_1, \ldots, y_n)}{x_1! \cdots x_k! \partial y_1^{x_1} \cdots \partial y_k^{x_k}}$$

evaluated at $y_1 = \cdots = y_k = 0, y_{k+1} = \cdots = y_n = 1$; similarly for other $\overline{x}$'s. It follows that $G(\overline{x}; y)$ has the required form. $\square$

A *merging scheme* on $(n_1, \ldots, n_r)$ is a partition $M = (\bigcup_{i=1}^r ([n_i] \times \{i\}), \sim)$ such that $|P \cap ([n_i] \times \{i\})| \leq 1$ for each part P and each i. Each splitting $v = v_1 v_2 \ldots v_r$ *defines* a merging scheme on $(\|v_1\|, \ldots, \|v_r\|)$ in which $(m_i, i) \sim (m_j, j)$ iff the m_ith symbol of v_i equals the m_jth symbol of v_j. (The m_ith symbol of v_i is the $a \in S(v_i)$ that turns in m_i when v_i is canonized.) In the other way, if $(v_1, \ldots, v_r)$ is an r-tuple of sequences and M is a merging scheme on $(\|v_1\|, \ldots, \|v_r\|)$, there is a unique canonical sequence $v = M(v_1, \ldots, v_r)$ that can be split in $w_1 w_2 \ldots w_r$ so that each w_i is equivalent to v_i and the merging scheme defined by the splitting equals M. (To obtain v, for each part P of M and each $(m_i, i) \in P$ replace the occurrences of the m_ith symbol in v_i by the common symbol x_P. Concatenate the resulting v_i and canonize.) For instance, if M partitions $\bigcup_{i=1}^3 ([2] \times \{i\})$ in $\{(1,1), (1,3)\}$, $\{(2,1), (1,2), (2,3)\}$, and $\{(2,2)\}$, then $M(bab, 5aa, 1155) = 1212331122$.

Clearly, $\|M(v_1, \ldots, v_r)\| = |M|$. Notice also that if M is defined by the splitting $v = v_1 v_2 \ldots v_r$ then $M(v_1, \ldots, v_r)$ is just the canonization of v.

Lemma 3.6 *Let $v = M(v_1, \ldots, v_r)$ and $w = M(w_1, \ldots, w_r)$, for the same merging scheme M.*

1. *Let u_i^v and u_i^w be subsequences of v_i and w_i such that, for each i, u_i^v and u_i^w are equivalent and of the same type. The subsequence u^v of v, which takes the same positions in v as are those taken by the u_i^v's in $v_1 v_2 \ldots v_r$, is equivalent to and of the same type as the analogous subsequence u^w of w.*

2. *Let I be an ideal. If $T(u, v_i) = T(u, w_i)$ for each $u \in I$ and each i, then $T(u, v) = T(u, w)$ for each $u \in I$.*

Proof. 1 is immediate. To prove 2, consider an $f \in T(u, v)$ for a $u \in I$. Injection f is the type of a u-copy t^v in v and t^v is composed from some subsequences t_i^v of v_i. By the assumption (each t_i^v is equivalent to some $s_i \in I$), there exist subsequences t_i^w of w_i which are equivalent to t_i^v and are of the same type. The subsequence t^w of w proves, by 1, that $f \in T(u, w)$ as well. The converse is proved similarly, so $T(u, v) = T(u, w)$. □

Notice that the lemma and the whole proof works even for $I \subset S^*$ closed only to *contiguous* subsequences (intervals).

Suppose $v \in S^*$, $X \subset S(v)$, $|X| \geq 2$, and $s \in \mathbf{N}$. In the (s, X)-*splitting* $v = v_1 v_2 \ldots v_r$, v_1 is the unique initial interval such that $|X \cap S(v_1)| = |X| - 1$ and the only symbol of X missing in v_1 appears immediately after v_1, v_2 is the unique interval following after v_1 with the same property and so on. The splitting is terminated if $X \not\subset S(w)$ for the residual interval w or if s intervals $v = v_1 v_2 \ldots v_{s-1} w$ have been already defined. Thus, $r \leq s$ and the splitting is unique.

Notice that if v and w are canonical and $v = v_1 v_2 \ldots v_r$ and $w = w_1 w_2 \ldots w_t$ define the same merging scheme (in particular, $r = t$), then the former splitting is the (s, X)-splitting of v if and only if the latter spliting is the (s, X)-splitting of w.

Lemma 3.7 *Suppose $v \in S^*$ is canonical, $u \in S^*$, and $f : [\|u\|] \to S(v)$ is an injection. Let $X = \mathrm{Im}(f)$, $v = v_1 v_2 \ldots v_r$ be the (s, X)-splitting, and $2|u| \leq s$. If there is a u-copy in v of type f that is contained in a single v_j then there is another u-copy of type f that is not contained in a single v_i.*

Proof. We can suppose that $u = a_1 a_2 \ldots a_t$ is canonical. If the assumption is fulfilled then, by the definition of (s, X)-splitting, inevitably $j = r = s$. But then, since $X \subset S(v_i v_{i+1})$ for each i, we choose an occurrence of $f(a_1)$ in v_1v_2, an occurrence of $f(a_2)$ in v_3v_4 etc. and obtain a u-copy of type f that is split into several v_i's. □

Suppose that J is an ideal and $n \geq m \geq 2, s > 0$ are integers. For every $v \in S^*, \|v\| = n$ we define a *color* C of v, which will be a triple determined uniquely by v, J, n, m, and s. For each X an m-subset of $S(v)$ we consider the $(2s, X)$-splitting $v = v_1^X v_2^X \dots v_{r(X)}^X$. Superposing all these $\binom{n}{m}$ splittings, we obtain a unique *superposed* splitting $v = v_1 v_2 \dots v_r$. Let M be the merging scheme defined by it. We define $n_i = \|v_i\|$ and $F_i \in \mathcal{F}(J, n_i)$ as having on $u \in J$ the value $T(u, v_i)$; notice that $n_1 = m - 1$ and $|M| = n$. The color of v is the triple $C = ((n_1, \dots, n_r), M, (F_1, \dots, F_r))$.

It is clear that $n_i \leq n$ (in fact, for $i < r$ even $n_i \leq m - 1$), $r \leq 2s\binom{n}{m}$, and $F_i \in \mathcal{F}(J, n_i)$. Thus — for given J, n, m, and s — the number of all possible colors is finite. Equivalent sequences have the same color. Let S_C^* be the set of all v with color C. The sets S_C^* are disjoint and their number is finite.

Now we perform the induction step. We are given an $n \in \mathbf{N}$, an ideal I that is different from $I(r)$ (case $I = I(r)$ was settled in Lemma 3.5), and a mapping $F \in \mathcal{F}(I, n)$. There is a $z \in I$ that is maximal (to $\prec$) and satisfies $\|z\| \geq 2$. Hence, $I \backslash \{z\}$ is an ideal for which Theorem 3.2 holds for any $n' \leq n$ and any $F' \in \mathcal{F}(I \backslash \{z\}, n')$. We set $J = I \backslash \{z\}$, $m = \|z\|$, $s = |z|$, and consider colors and sets S_C^* corresponding to these J, n, m, and s. (We can assume that $n \geq m$, otherwise we are done.)

Lemma 3.8 *If $w_1, w_2 \in S_C^*$ then $T(u, w_1) = T(u, w_2)$ for each $u \in I$.*

Proof. The claim follows at once from 2 of Lemma 3.6 if $u \in J$. It remains to verify it for $u = z$. W.l.o.g., w_1 and w_2 are canonical. Consider any $f \in T(z, w_1)$. We claim that there is always a z-copy in w_1 of type f that is split into several intervals in the the superposed splitting; the pieces must be then equivalent to sequences in J. By Lemma 3.7, there is even such a copy that is split already in the $(2s, X)$-splitting of w_1 with $X = \mathrm{Im}(f)$. By the definition of color and by 1 of Lemma 3.6, $f \in T(z, w_2)$. The converse is proved similarly, so $T(z, w_1) = T(z, w_2)$. □

Lemma 3.9 *The canonical sequences $v \in S_C^*$, where $C = ((n_1, \dots, n_r), M, (F_1, \dots, F_r))$, are in bijection with the r-tuples $(w_1, \dots, w_r)$ of canonical sequences satisfying $\|w_i\| = n_i$, $T(u, w_i) = F_i(u)$ for each $u \in J$, and $|v| = |w_1| + \dots + |w_r|$.*

Proof. Each canonical $v \in S_C^*$ is sent to $(v_1^c, \ldots, v_r^c)$, where v_i^c is the canonized ith interval of the superposed splitting of v. In the other way, $(w_1, \ldots, w_r)$ is sent to $v = M(w_1, \ldots, w_r)$. By the paragraphs before Lemmas 3.6 and 3.7, both correspondences are inverses of one another. (More precisely, we use that the remark before Lemma 3.7 applies also to the superposed splittings.) $\square$

Finally, let $\mathcal{G}$ be the set of all colors C for which the mapping sending $u \in I$ to $T(u, v)$, where $v \in S_C^*$ is arbitrary (by Lemma 3.8 this makes sense), equals the prescribed mapping F. Let $G(n, I, F; y)$ be the GF introduced in Theorem 3.2. By Lemma 3.9,

$$G(n, I, F; y) = \sum_{C \in \mathcal{G}} G(n_1, J, F_1; y) G(n_2, J, F_2; y) \cdots G(n_r, J, F_r; y).$$

By the induction hypothesis on $G(n_i, J, F_i; y)$, $G(n, I, F; y)$ is as stated. This finishes the proof of Theorem 3.2.

4 One doubleton

In Sections 4 and 5 n is not restricted. By Examples 2 and 3, in general we cannot expect $G(u; 1, y)$ be rational if the srp u has more than one doubleton. To complement this, we sketch the proof of the following result.

Theorem 4.1 *If u is a srp with at most one doubleton then $G(u; 1, y) \in \mathbf{Z}(y)$.*

If u has only singletons, the GF is rational by Example 4. Srp with one doubleton has the form $u(r, s, t) = a_1 \ldots a_r b a_{r+1} \ldots a_{r+s} b a_{r+s+1} \ldots a_{r+s+t}$, for some distinct $a_i, b \in S$ and $0 \leq r, s, t$. First we indicate the proof for the case $r = t = 0$. Then we describe how the full result can be proved using that case and a refinement of Theorem 3.1. In Example 6 we calculate the GF for $u(0, 2, 0)$.

Let $u(s) = u(0, s, 0) = ab_1 \ldots b_s a$. For $v \in S^*$, $E(v)$ denotes the subsequence of v that consists of the first and last appearances of all $a \in S(v)$.

Lemma 4.2 *If $u(s) \prec v$ then $u(s) \prec E(v)$.*

Proof. Let $a_1 = a_2 = a$ be the first and last term of a $u(s)$-copy in v and $X \subset S(v), a \notin X, |X| = s$ be the set of some s symbols appearing between a_1 and a_2. We can assume that both a_i lie in $E(v)$. Let $Y \subset X$ be the symbols that have neither the first nor the last appearance between a_1 and a_2. If $Y = \emptyset$ we easily form a $u(s)$-copy lying in $E(v)$. Otherwise let $b \in Y$ have the earliest first appearance of all $x \in Y$. The first and last appearance of b, the first appearances of $x \in Y \backslash \{b\}$, a_1, and first or last appearance of each $x \in X \backslash Y$ (the one lying between a_1 and a_2) form a $u(s)$-copy in $E(v)$. $\Box$

Suppose $v \not\succ u(s)$ and consider the $(s+1)$-splitting $v = v_1 v_2 \dots v_t$. Clearly, $S(v_i) \cap S(v_j) = \emptyset$ whenever $j - i > 1$. Let $w = w_1 w_2 \dots w_t$ where $w_i = v_i \cap E(v)$. Note that (i) there is only a finite number of possibilities for w_i's, (ii) v can be obtained back from w by filling the gaps in w arbitrarily (Lemma 4.2), and (iii) the admissible w's are determined only by some local restrictions on the consecutive pairs $w_i w_{i+1}$. By the transfer matrix method (see Chapter 4 of [26]), $G(u(s); 1, y)$ is a rational function.

For the full Theorem 4.1 we need a variant of Theorem 3.1. Let $n \in \mathbf{N}$ and $z \in S^*$ be such that $z \not\succ aaa$ and $\|z\| = n$. Let $P(R, z; n, l)$ count the canonical v such that $\|v\| = n$, $|v| = l$, $v \not\succ u$ for each $u \in R$, and $E(v)$ is equivalent to z. Modifying the proof in Section 3, we can prove a refinement of Theorem 3.1 with $P(R; n, l)$ replaced by $P(R, z; n, l)$.

Suppose $v \not\succ u(r, s, t)$. The *end* symbols $x \in S(v)$ are the 1st, 2nd, …, and $(r + t)$th symbol of v and of the reversed v; we have $\leq 2(r + t)$ end symbols. The other symbols are called *middle* symbols. Let w be the subsequence of v formed only by the middle symbols. Clearly, $w \not\succ u(s)$. Let $w = w_1 w_2 \dots w_j$ be the $(s + 1)$-splitting and v_i be the interval of v spanned by w_i. If no end symbol appears in v_i, we call it *pure*; then $v_i = w_i$. The number of nonpure v_i's is $\leq n_0 = n_0(r + s + t)$. For an $n_1 > 0$ we add to each nonpure v_i n_1 neighbouring (possibly pure) v_k's and obtain this way a subsequence v' of v with these properties: (i) $\|v'\| \leq n_2$ and (ii) the ways in which v' can be extended to v by adding pure v_i's depend only on $E(v')$. Given $E(v')$, the extensions can be counted as in the $r = t = 0$ case and the corresponding GF is rational. The GF counting v''s with a fixed $E(v')$ is also rational, by the refinement of Theorem 3.1 used with $R = \{u(r, s, t)\}$. Summing the products over all possible $E(v')$'s we infer that $G(u(r, s, t); 1, y) \in \mathbf{Z}(y)$.

EXAMPLE 6. We calculate $G(u(2); 1, y) = G(abca; 1, y)$. Let $v, v \not\succ abca$ be canonical and irreducible, that is $v = v_1 v_2$ with $S(v_1) \cap S(v_2) = \emptyset$ implies $v_1 = \emptyset$ or $v_2 = \emptyset$. If $G_I(y)$ is the GF counting such v's, then $G(abca; 1, y) = G_I(y)/(1 - G_I(y))$. It is easy to verify that such v's are the sequences in $\{1, 2\}^*$ starting with 1 and distinct from $11 \dots 122 \dots 2$. We have

$2^{l-1} - l + 1$ of them of length l and $G_I(y) = y(1-3y+3y^2)(1-2y)^{-1}(1-y)^{-2}$. Thus,

$$G(abca; 1, y) = \frac{y(1-3y+3y^2)}{1-5y+8y^2-5y^3}.$$

We have also determined $G(u(3); 1, y)$:

$$G(abcda; 1, y) = \frac{y(1-11y+49y^2-112y^3+138y^4-87y^5+20y^6)}{1-13y+70y^2-202y^3+336y^4-321y^5+163y^6-32y^7}.$$

We leave the verification of the formula to the interested reader as an exercise.

5 Fixed number of crossings

Bóna [6] proved that the GF counting partitions with a fixed number of *abab*-copies belongs to $\mathbf{Z}(\sqrt{1-4y})$. We show that the same is true for partitions with a fixed number of pairs of crossing parts. The *crossing graph* $\mathcal{G}(u)$ of $u = ([l], \sim)$ has parts of u as vertices and $\{P, Q\}$ is an edge iff there is an *abab*-copy lying in $P \cup Q$.

Theorem 5.1 *For each k the GF*

$$G(k; y) = \sum_{l \geq 1} \#\{u = ([l], \sim) : \ \mathcal{G}(u) \text{ has } k \text{ edges}\} \cdot y^l$$

belongs to $\mathbf{Z}(\sqrt{1-4y})$.

In particular, the numbers of partitions in question form a P-recursive sequence; see [27] for more information on P-recursiveness.

The proof is based on two lemmas. The first lemma is a part of folklore and its easy proof is omitted.

Lemma 5.2 *Let $A, B \subset V(\mathcal{G}(u))$ be two distinct components of $\mathcal{G}(u)$. Then one of the sets $\bigcup A$ and $\bigcup B$ (subsets of $[l]$) precedes the other or one of them is contained in a gap of the other.*

If $\bigcup A$ is contained in a gap of $\bigcup B$ we say that B *covers* A.

Lemma 5.3 *For each k the GF*

$$G(c, k; y) = \sum_{l \geq 1} \#\{u = ([l], \sim) : \ \mathcal{G}(u) \text{ is connected and has } k \text{ edges}\} \cdot y^l$$

belongs to $\mathbf{Z}(y)$.

Proof. The partitions involved have at most $k+1$ parts. The proof follows from Theorem 3.2 by setting $R = \{abab\}$ and summing all cases. $\square$

Let $C_j(y)$ be the GF counting by $|u|$ the pairs $(u, (i_1, \ldots, i_j))$ where u is a noncrossing partition, $0 \le i_1 \le i_2 \le \cdots \le i_j \le |u|$, and $u = \emptyset$ is allowed. Thus, $C_0(y) = 1 + G(abab; 1, y)$ is given in Example 2 and $C_1(y) = yC_0'(y) + C_0(y)$. Since $C_j(y)$ expresses in terms of derivatives of $C_0(y)$, $C_j(y) \in \mathbf{Z}(\sqrt{1-4y})$ for each j. Similarly, let $G_j(c, k; y)$ count the pairs $(u, (i_1, \ldots, i_j))$ where $\mathcal{G}(u)$ is connected and has k edges and $1 \le i_1 < \cdots < i_j < |u|$. Using derivatives and Lemma 5.3, we see that $G_j(c, k; y) \in \mathbf{Z}(y)$ for each j.

Consider a u and the graph $\mathcal{G}(u)$. Components distinct from isolated vertices are the *nontrivial* components. The *top* components are the nontrivial components that are not covered by any nontrivial component. Let X be the set of the isolated vertices that are not covered by any nontrivial component. By Lemma 5.2, u has the following structure.

Some of the sets $\bigcup A_i$, where $A_1, \ldots, A_m$ are the top components listed so that $\bigcup A_i$ precedes $\bigcup A_{i+1}$, are inserted in (not necessarily distinct) gaps of the noncrossing partition $\bigcup X$ and the remaining ones precede $\bigcup X$ or follow after it. Suppose A_i spans $k_i^0 > 0$ edges. $\bigcup A_i$ has $r(i) \ge 0$ *special* gaps each of which contains a subgraph spanning $k_i^j > 0$ edges, $j = 1, \ldots, r(i)$ (we list the gaps from left to right). The remaining gaps contain only isolated vertices, i.e. a noncrossing partition. Each component of $\mathcal{G}(u)$ not in $\{A_1, \ldots, A_m\} \cup X$ is covered by an A_i and lies in a special gap if it is nontrivial.

We prove Theorem 5.1 by induction on k. For $k = 0$ it holds because $G(0; y) = (1 - 2y - (1-4y)^{1/2})/(2y)$, see Example 2. Suppose that $k > 0$ and the theorem holds for each smaller k'. The problem breaks in finitely many disjoint cases according to the tuples $(k_1^0, \ldots, k_1^{r(1)}; \ldots; k_m^0, \ldots, k_m^{r(m)})$, $m \ge 1$, $r(i) \ge 0$, $k_i^j > 0$, $\sum_{i,j} k_i^j = k$. Let us consider the GF for one case.

The positions of $\bigcup A_i$'s with respect to $\bigcup X$ are counted by $C_m(y)$ and the positions of the special gaps of $\bigcup A_i$ are counted by $G_{r(i)}(c, k_i^0; y)$. The content of a gap of $\bigcup A_i$ is counted by $G(k_i^j; y)$ if it is special and by $C_0(y)$ otherwise.

So the total GF equals

$$\sum C_m(y) \prod_{i=1}^{m} \frac{G_{r(i)}(c, k_i^0; yC_0(y)) \cdot G(k_i^1; y) \cdot \ldots \cdot G(k_i^{r(i)}; y)}{C_0(y)^{r(i)+1}},$$

where we sum over all cases. By the above remarks, $G_{r(i)}(c, k_i^0; y) \in \mathbf{Z}(y)$ and $C_0(y), C_m(y) \in \mathbf{Z}(\sqrt{1-4y})$. $G(k_i^j; y) \in \mathbf{Z}(\sqrt{1-4y})$ by the induction hypothesis. Hence, the total GF belongs to $\mathbf{Z}(\sqrt{1-4y})$.

6 Concluding remarks

Recently, Alon and Friedgut [1] applied extremal methods to forbidden permutations. Using results on generalized Davenport–Schinzel sequences, they gave an almost exponential upper bound to $S_n(p)$ for each p and they extended the class of p with known exponential upper bound.

We conclude by proposing few problems. PROBLEM 1. Prove (or disprove) the conjecture given in Section 1: $P(u_p;\cdot,l) = O(c^l)$ for each permutation p. PROBLEM 2. The asymptotics of $S_n(12\ldots m)$ was found by Regev [19]. What is the asymptotics of $P(12\ldots m12\ldots m;\cdot,l)$ and $P(12\ldots mm\ldots 21;\cdot,l)$? Case $m=2$ is settled in Examples 2 and 3. PROBLEM 3. Find $G(u;1,y)$ for a srp u with more than two doubletons, e.g. for $u = abcabc$ or $u = abcbca$. PROBLEM 4. Characterize $G(u;1,y)$ for srps with two doubletons. Does the GF always satisfy a quadratic equation? PROBLEM 5. Recall that $u(s) = ab_1b_2\ldots b_sa$. What can be said about the rational function $G(u(s);1,y)$? Let $c_s = \lim_{l\to\infty} P(u(s);\cdot,l)^{1/l}$; thus $c_1 = 2$, $c_2 = 2.75488\ldots, c_3 = 3.46357\ldots$, see Example 6. What is the behaviour of c_s for $s \to \infty$? PROBLEM 6. What changes in Section 5 when *abab* is replaced by *abba*? PROBLEM 7. Gessel mentions [9] the conjecture that $\{S_n(p)\}_{n\geq 1}$ is always P-recursive. Prove (or disprove) that for each u the numbers $\{P(u;\cdot,l)\}_{l\geq 1}$ form a P-recursive sequence. Here u is any partition, cf. Example 1. Note that unlike $\{n!\}_{n\geq 1}$ the sequence of Bell numbers $\{P(\emptyset;\cdot,l)\}_{l\geq 1}$ is not P-recursive.

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