Alimov's theorem: any ordered semigroup without infinitesimals is commutative

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Among other results, Alimov [2] proved the following interesting theorem.

Theorem (Alimov, 1950). *Every ordered semigroup $A = (A, +, <)$ with no pair of infinitesimally close elements is commutative.*

In his theorem, $+ : A \times A \to A$ is an associative binary operation — for every $a, b, c \in A$ one has $a + (b + c) = (a + b) + c$ — $< \subset A \times A$ is a transitive and trichotomic binary relation — for every $a, b, c \in A$ one has $a < b < c \Rightarrow a < c$ and exactly one of $a < b$, $a = b$, and $b < a$ — and $+$ is monotonous to $<$ in the sense that for every $a, b, c \in A$ one has $a < b \Rightarrow c + a < c + b$, $a + c < b + c$. One does not assume commutativity of $+$ and the theorem says that it is forced, under the stated assumption of absence of pairs $a, b \in A$ such that ($\mathbb{N} = \{1, 2, \dots\}$)

$$\forall\, n \in \mathbb{N} :\ na < nb < (n + 1)a \quad \text{or} \quad \forall\, n \in \mathbb{N} :\ na > nb > (n + 1)a\ .$$

Here na abbreviates $a + a + \cdots + a$ with n summands. Such a pair of elements $a, b \in A$ is called an *anomalous pair*, and A is *non-anomalous* if it has none. The theorem thus says that every non-anomalous ordered semigroup is commutative.

An obvious interpretation of an anomalous pair, satisfying for example the first system of inequalities, is that $(0 <)\ a < b$ but b is larger than a only by an infinitesimal amount that cannot be magnified to exceed a by multiplication with any $n \in \mathbb{N}$, no matter how big. In the ordered semigroup $(\mathbb{R}, +, <)$ with the ordinary addition and comparison of real numbers clearly no anomalous pair exists because $0 < a < b$ implies that $nb > (n + 1)a$ whenever $n > a/(b - a)$. And, of course, the ordinary addition of reals is commutative. Thus the theorem shows that commutativity of addition of real numbers follows from the axioms of an ordered, and generally non-commutative, semigroup and the absence of infinitesimally small positive elements. On the other hand, the non-commutative ordered semigroup $(\{a, b\}^*, +, <)$, where $\{a, b\}^*$ is the set of words over the two-element alphabet $\{a, b\}$, $+$ is concatenation of words, and $<$ is the comparison

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first by length and then, for words with equal lengths, lexicographically (from the left and setting $a < b$), has anomalous pairs: a, b is one as

$$a < b < aa < bb < aaa < bbb < aaaa < \dots \ .$$

We learned Alimov's theorem from the interesting preprint of Binder [4] in which he constructs $\mathbb{R}$ as the terminal object in the category of pointed non-anomalous ordered semigroups. Algebra contains a number of results asserting commutativity of an apriori possibly non-commutative operation, caused by its interplay with other operation(s) or, as in *Alimov's theorem* which is perhaps the simplest example of these results, relation(s).

- Well known is *Wedderburn's theorem*: every finite division ring (algebra) $(R, +, \cdot)$ is commutative. It should be properly called the *Maclagan-Wedderburn–Dickson theorem* (note the hyphens) because Wedderburn was actually Joseph H. Maclagan-Wedderburn, his proof [13] contained a gap, and the first correct proof appears to be that of Dickson [8]. A short proof based on cyclotomic polynomials was found by Witt [15]. Adam and Mutschler [1] provide interesting material on Wedderburn's original proof and history of Wedderburn's theorem; we draw information and references from their preprint. Another curiosity is Ted Kaczynski's publication [11] on the topic, see [1] for an annotation of his work.

- *Jacobson's theorem* [10] says that a ring $R = (R, +, \cdot)$ is commutative if for every $x \in R$ there is an $n \in \mathbb{N}$ such that $n \geq 2$ and $x^n = x$. This clearly holds in a finite division ring and we have therefore a generalization of Wedderburn's theorem. Another theorem of this type, taken from the survey article of Pinter-Lucke [14] that lists many more such results, is the *theorem of Bell* [3]: R is commutative if and only if for every $x, y \in R$ there exist $m, n \in \mathbb{N}$ with $xy = y^m x^n$.

- The *Eckmann–Hilton argument* [9, 16] concerns two binary operations $+$ and $\times$ on the same set A that are unital ($0 \in A$ exists such that $a + 0 = 0 + a = a$ for every $a \in A$ and similarly for $\times$) and mutual homomorphisms ($(a+b)\times(c+d) = (a\times c)+(b\times d)$ for every $a, b, c, d \in A$). Their interplay forces that they coincide, $+ = \times$, are commutative and associative. See Kock [12] and Bremner and Madariaga [7] for more results on this algebraic structure of double semigroups.

- *An abelian variety is commutative* (stated in Bombieri and Gubler [6, Corollary 8.2.10] but who did prove it first?). An abelian variety is a geometrically irreducible and geometrically reduced complete group variety. A group variety is an apriori possibly non-commutative group that is also a variety and the group operation and inverse are morphisms. See [6] for more details and further unfolding of the terminology and definitions.

- In their interesting preprint *Blasiak and Fomin* [5] "study the phenomenon in which commutation relations for sequences of elements in a ring are implied by similar relations for subsequences involving at most three indices at a time." For example, they prove that if $g_1, \dots, g_n, h_1, \dots, h_n$ are invertible elements in a ring then for every m-tuple $1 \le s_1 < s_2 < \dots < s_m \le n$ the product $g_{s_m} \dots g_1$ commutes with both $h_{s_m} \dots h_1$ and $h_{s_m} + \dots + h_1$ $\iff$ this holds for any $m \le 3$.

The proof of Alimov's theorem (after [2])

Let $A = (A, +, <)$ be an ordered semigroup. We do not assume commutativity of $+$ but eventually deduce it for non-anomalous A. Trichotomy of $<$ and monotonicity of $+$ imply the cancellation law: for every $a, b, c \in A$ if $a+c < b+c$ then $a < b$, and the same for $<$ replaced with $=$ and for exchanged summands.

Let $a, b \in A$ be arbitrary, then exactly one of $b + a > b$, $b + a = b$, and $b+a < b$ occurs. In the first case when $b+a > b$ monotonicity and associativity of $+$ and the cancellation law imply that for every $c \in A$,

$$b + (a + c) = (b + a) + c > b + c \rightsquigarrow a + c > c \; .$$

In the other two cases we get similarly that $a + c = c$, respectively $a + c < c$, for every $c \in A$. Hence for every $a \in A$, exactly one of the three cases occurs: $a + c > c$ for every $c \in A$, $a + c = c$ for every $c \in A$, and $a + c < c$ for every $c \in A$. In the first case we say that a is *positive*, in the second we call it a *zero element*, and in the third we say that a is *negative*.

Thus A partitions into negative elements, zero elements, and positive elements; these sets may be empty. We defined this partition by adding a from the left but it follows from the beginning of the argument that addition of a from the right gives the same result, the same partition. In particular, if $a, b \in A$ are two zero elements then $a + b = a$ and $a + b = b$, hence $a = b$. Thus A has at most one zero element, which we then denote as $0 \in A$. If A has no zero element, for simplicity we add it to A. It follows that $a \in A$ is negative if $a < 0$ and positive if $a > 0$ (and zero element if $a = 0$).

We start the proper proof of Alimov's theorem. We assume that A is non-anomalous, take any two elements $a, b \in A$, and prove that

$$a + b = b + a \; .$$

If $a = 0$ or $b = 0$ then it clearly holds. Thus we need to distinguish three cases, (i) $a, b > 0$, (ii) $a, b < 0$, and (iii) $a < 0 < b$.

Let (i) occur and a, b be positive. We show that if $a+b \ne b+a$ then $a+b, b+a$ is an anomalous pair. Indeed, then we may assume that $a + b < b + a$ and for every $n \in \mathbb{N}$ get

$$(n + 1)(a + b) = a + n(b + a) + b > n(b + a) + b > n(b + a) > n(a + b)$$

— the first $=$ is by associativity, the second $>$ is by positivity of a, the third $>$ is by positivity of b, and the fourth $>$ is by the assumption that $a + b < b + a$ and monotonicity of addition ($r, s, t, u \in A$ with $r < s, t < u$ gives $r+t < s+u$). So $a + b = b + a$

The case (ii) with both a, b negative is treated as (i), we only start with $a + b > b + a$ and reverse the inequalities in the last displayed calculation.

Let the case (iii) occur with a negative and b positive. Now we have three subcases: (a) $a + b = 0$, (b) $a + b > 0$, and (c) $a + b < 0$. In the subcase (a) we get by associativity $a + (b + a) = a$ and so $b + a = 0$ and $a + b = b + a$.

In the subcase (b) we have $b, a + b > 0$. If $a + b \neq b + a$, say $a + b < b + a$, we get the contradiction

$$\begin{aligned} 2(b + a) &= (b + (a + b)) + a = ((a + b) + b) + a = (a + b) + (b + a) \\ &< (b + a) + (b + a) = 2(b + a) \end{aligned}$$

— by associativity, the case (i) applied to $b, a+b$, associativity, and monotonicity and the assumption that $a + b < b + a$. If $a + b > b + a$, we get a similar contradiction, only the last inequality in the calculation gets reversed. We see that in the subcase (b) we have $a + b = b + a$.

We consider the last subcase (c) of the case (iii). Now $a + b, a < 0$. If $a + b \neq b + a$, say $a + b < b + a$, we get similarly to the subcase (b) the contradiction

$$\begin{aligned} 2(b + a) &= b + ((a + b) + a) = b + (a + (a + b)) = (b + a) + (a + b) \\ &< (b + a) + (b + a) = 2(b + a)\ , \end{aligned}$$

and similarly if $a + b > b + a$. Thus also in the final subcase (c) we have $a + b = b + a$ and are done. $\square$

The above proof is a slight simplification of that in [2, p. 573]. In the subcase (b) of the case (iii) when b and $a + b$ are positive, Alimov first derives that also $b + a$ is positive and only then obtains a contradiction similar to ours. This detour is unnecessary and it suffices to know just that $b, a + b > 0$.

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