(L5) (March 27, 2020) | In L3 I promised (1)

two applications of Abel's summation and I showed so far just one, the deduction of the 2nd Mertens formula ($\sum_{p\le x}\frac{1}{p}=\log\log x+c+O(\frac{1}{\log x})$) from the 1st one ($\sum_{p\le x}\frac{\log p}{p}=\log x+O(1)$). The second application is the following result. Recall that $\pi(x)$ is the number of primes $\le x$ ($x\in\mathbb{R}$).

Theorem (P.L. Čebyšev, $\doteq 1850$) If the limit

$$\alpha=\lim_{x\to+\infty}\frac{\pi(x)}{x/\log x}$$

exists then $\alpha=1$.

Proof. We prove that $\liminf_{x\to+\infty}\frac{\pi(x)}{x/\log x}\le 1\le$ $\le\limsup_{x\to+\infty}\frac{\pi(x)}{x/\log x}$ (which suffices). Suppose for $\lightning$ that the lim inf is >1; for the lim sup we argue similarly. So $\exists\,\alpha>1$ and $\exists\,t_0>0$ s.t. $\pi(x)>\frac{\alpha x}{\log x}$ whenever $x>t_0$. We use Abel's summation for the interval $[t_0,x]$ ($x>t_0$), function $f(t)=\frac{1}{t}$ and sequence (a_n) being the char. f. of primes, $a_n = \begin{cases} 1 \cdots n = p \\ 0 \cdots n \neq p \end{cases}$:

$$\sum_{x_0 < n \leq x} a_n f(n) = \sum_{x_0 < p \leq x} \frac{1}{p} = \frac{\pi(x)}{x} - \frac{\pi(x_0)}{x_0} + \underbrace{\int_{x_0}^{x} \frac{\pi(t)}{t^2}}_{>1!} \#$$

$$\# > O(1) + \alpha \int_{x_0}^{x} \frac{dt}{t \log t} = \alpha \log\log x + O(1).$$

But this is a contradiction with the 2nd M. formula $\boxtimes$

Similarly one can prove by Abel's summation that the asymptotics like

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^3(x)}\right) \text{ does } \underline{\text{not}} \text{ hold}$$

- by the (strong form of the) PNT (prime number theorem) the correct main term in the asymptotics of $\pi(x)$ is the logarithmic integral ~~Li~~ $Li(x) := \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$. In the 2nd lecture I promised 5 theorems on $\sum$s and $\int$s and so far we have seen 4 of them. There are many interesting identities ~~between~~ relating $\sum$s and $\int$s. I mention without

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proofs two of them. The first is an alternative to the Euler-Maclaurin summation formula due to I. Pinelis in 2015 (see arXiv: 1511.03247).

It approximates $\sum f(n)$ only by $\int f$'s, but ~~the~~ a derivative~~s~~ of f ~~are~~ is still hidden in the error term.

Explicitly: if $n, m \in \mathbb{N}$ and $f \in C^{2m}(0,n)$ then $\sum_{q=0}^{n-1} f(q) = A_m - R_m$ where, with

$$\gamma_{m,j} = (-1)^{j-1} \frac{2}{j} \binom{2m}{m+j} \cdot \frac{1}{\binom{2m}{m}},$$

$$A_m = \sum_{j=1}^{m} \gamma_{m,j} \sum_{i=0}^{j-1} \int_{i-\frac{j}{2}}^{n-1+\frac{j}{2}-i} f \text{ and}$$

$$R_m = \frac{1}{(2m-1)!\,2^{2m+1}} \int_0^1 \left((1-s)^{2m-1} \int_{-1}^{1} \sum_{j=1}^{m} \gamma_{m,j}\, j^{2m+1} \sum_{q=0}^{n-1} f^{(2m)}\left(q + \frac{jsv}{2}\right) dv \right) ds.$$

One can bound $|R_m|$ nicely, see the paper.

I give as the 5th theorem an identity due basically to S. D. Poisson (1781 – 1840) who was a French physicist and mathem.

Let $L^1(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{R} \mid \int_{-\infty}^{+\infty} |f| < +\infty \}$ and $\hat{f}(x) := \int_{-\infty}^{+\infty} f(t) e^{-2\pi i \cdot x t} \, dt$ be the <u>Fourier transform</u> of f, $\hat{f} : \mathbb{R} \to \mathbb{R}$ (exercise: prove that indeed $\overline{\hat{f}(x)} = \hat{f}(x)$), for $f \in L^1(\mathbb{R})$. One precise form of the <u>Poisson summation formula</u>), which roughly says that $\sum_{n=-\infty}^{+\infty} f(n) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)$, is the following.

$\boxed{\text{Theorem 5 (Poisson } \Sigma \text{ formula)}}$ If $f \in L^1(\mathbb{R})$ and f is continuous (on $\mathbb{R}$) then

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n \in \mathbb{Z}} f(n) = \lim_{x \to +\infty} \sum_{-x < n < x} \hat{f}(n).$$

(Here we have absolutely convergent series.) There are other variants of the P. Σ formula.

That was the 1st topic, applications of real analysis, more precisely of identities relating

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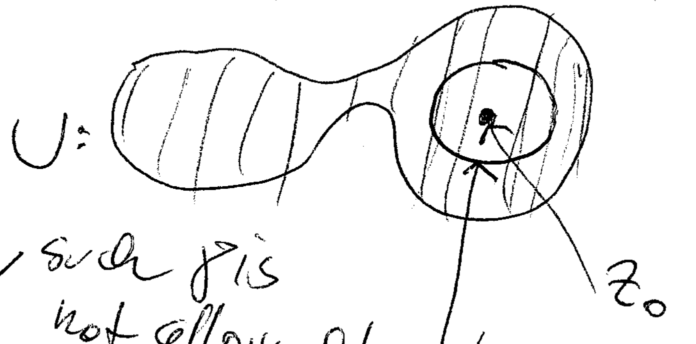
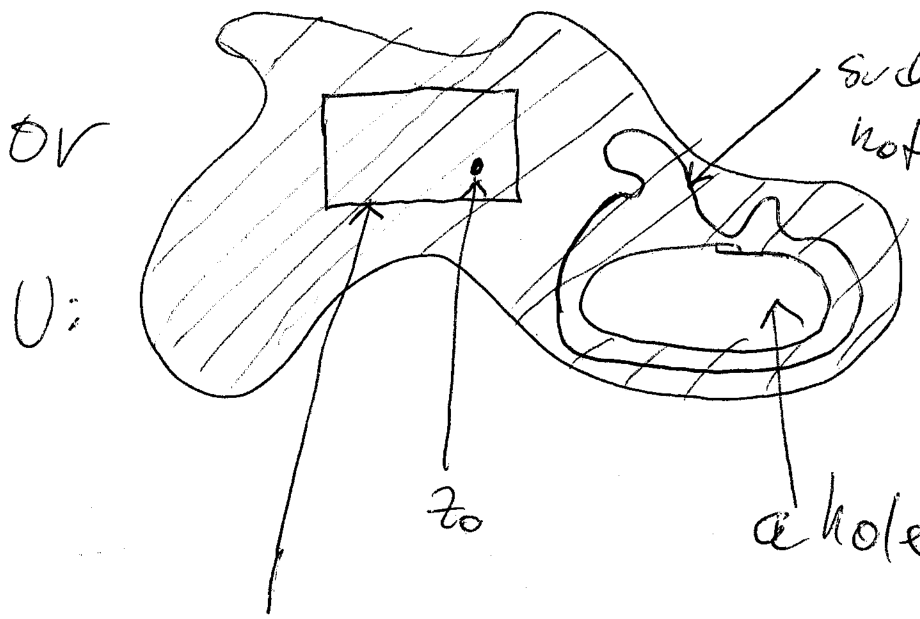
Σs and $\int$s, in analytic NT. The 2nd topic is applications of complex analysis, more precisely of the <u>Cauchy formula</u>: let $U \subset \mathbb{C}$, $U \neq \emptyset$, be an open set, $z_0 \in U$ be a point, $f: U \to \mathbb{C}$ be a <u>holomorphic function</u> — which means that for every point $u \in U$ the limit $\lim_{z \to u} \frac{f(z)-f(u)}{z-u} =: f'(u) \in \mathbb{C}$ exists — and $\gamma: [a,b] \to U$ be a <u>piecewise smooth simple closed curve</u> — γ is continuous, γ' is continuous except for possibly finitely many points t_i, $a \leq t_1 < t_2 < \dots < t_k \leq b$, the one-sided limits $\gamma'(t_i - 0)$ and $\gamma'(t_i + 0)$ exist and are finite, $\gamma(a) = \gamma(b)$ (γ is closed), and γ is injective (γ is simple, does not self-intersect) — s.t. $z_0 \in \underbrace{\text{Interior of } \gamma \subset U}_{(*)}$, then we have for $f(z_0)$ the Cauchy formula

$$\boxed{f(z_0) = \int_\gamma \frac{f(z)\,dz}{z-z_0}}.$$

So γ may look like



or



$\gamma([a,b])$ is a rectangle (γ' does not exist in the corners). But what is $\int_\gamma \ldots$? If $g: U \to \mathbb{C}$ is a function and $\gamma: [a,b] \to U$ is a piece-wise smooth curve, then

$$\int_\gamma g := (R)\int_a^b \mathrm{Re}\,(g(\gamma(t))\cdot\gamma'(t))\,dt + i\cdot(R)\int_a^b \mathrm{Im}\,(g(\gamma(t))\cdot\gamma'(t))\,dt.$$

((R) – Riemann integrals) The last missing point is a precise definition of the Interior of γ. This is complicated in general but for circles or rectangles is clear. We leave it at that