(L 3) (1st coronavirus lecture, March 13, 2020) Recall that for real

$a < b$ and $f \in C^1(a,b)$ we have the identity: ($\boxed{\text{Thm. 2}}$)

$$\sum_{a < n \leq b} f(n) = \int_a^b f + \int_a^b (t - \lfloor t \rfloor) f'(t)\, dt + (a - \lfloor a \rfloor) f(a) -$$
$$- (b - \lfloor b \rfloor) f(b)$$

$\boxed{\{x\} := x - \lfloor x \rfloor \in [0,1)}$

where $\lfloor x \rfloor = \max(\{n \in \mathbb{Z} \mid n \leq x\})$ is the lower integer part of $x \in \mathbb{R}$. And $\{x\}$ is the fractional part.

What does it give for $f(x) = \frac{1}{x}$?

$$\sum_{k=2}^{n} \frac{1}{k} = \int_1^n \frac{1}{x} - \int_1^n \frac{x - \lfloor x \rfloor}{x^2} + 0 - 0 = \log n - \int_1^{+\infty} \frac{x - \lfloor x \rfloor}{x^2} +$$
$$+ \int_n^{+\infty} \frac{x - \lfloor x \rfloor}{x^2} = \log n - c + O(1/n) \text{ where } c > 0 \text{ is a const..}$$

this is because the integrand in the two $\int$s is $O(1/x^2)$ (*) for $x \in [1, +\infty)$. We have thus proven the following:

$\boxed{\text{Theorem}}$ For $n \in \mathbb{N}$ the n-th harmonic number $H_n := \sum_{k=1}^{n} \frac{1}{k}$ has asymptotics

$$\boxed{H_n = \log n + \gamma + O\left(\frac{1}{n}\right)}$$ where

$$\gamma = 1 - \int_1^{+\infty} \frac{x - \lfloor x \rfloor}{x^2}\, dx \left(= \lim_{n \to \infty} (H_n - \log n) \right) = 0.577215\ldots$$

is the Euler (- Mascheroni) constant. Another formula for γ is $\gamma = -\int_0^1 \log\left(\log\left(\frac{1}{x}\right)\right) dx$.

(*) And $\int \frac{1}{x^2} = -\frac{1}{x}$. $(+c)$

$\boxed{\text{Problem, so far unsolved}}$ Is $\gamma \in \mathbb{R} \setminus \mathbb{Q}$???

A much easier $\boxed{\text{exercise 1}}$ for you: Prove that for $\forall$ $n \in \mathbb{N} \setminus \{1\}$, $H_n \notin \mathbb{N}$. That is, except $H_1 = 1$ no harmonic number is an integer. Here is one application of γ and of the previous asymptotics for H_n in elementary NT.

$\boxed{\text{Theorem (P. Dirichlet, 1849)}}$ Let $\tau(n) := \sum_{d \mid n} 1$ be the number of divisors of $n \in \mathbb{N}$. Then for every real $x \geq 1$,

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Proof. Let $x \geq 1$, $S = \{(a,b) \in \mathbb{N}^2 \mid ab \leq x\}$, $m = \lfloor \sqrt{x} \rfloor$, $T = \{(a,b) \in S \mid a \leq m\}$ and $U = \{(a,b) \in S \mid b \leq m\}$.

Then $|S| = \sum_{n \leq x} \tau(n)$, by grouping the $(a,b) \in S$ with a common value of the product ab. Also, $S = T \cup U$ because if $a, b \in \mathbb{N}$ and $ab \leq x$ then $a \leq \lfloor \sqrt{x} \rfloor$ or $b \leq \lfloor \sqrt{x} \rfloor$. ~~and~~

Further, $|T| = |U|$ (by the symmetry $a \leftrightarrow b$:

So $\sum_{n \leq x} \tau(n) = |S| = |T \cup U| = |T| + |U| - |T \cap U| =$

$$= 2|T| - \underbrace{|\{(a,b) \in \mathbb{N}^2 \mid a, b \leq m\}|}_{= m^2} = 2 \sum_{n \leq m} \left\lfloor \frac{x}{n} \right\rfloor - \lfloor \sqrt{x} \rfloor^2$$

$$= 2xH_m + O(m) - (\sqrt{x} - \underbrace{O(1)}_{\{\sqrt{x}\}})^2 = 2xH_m - x + O(m) =$$

$$= 2x\left(\log m + \gamma + O(1/m)\right) - x + O(m)$$

$$= 2x\log m + (2\gamma-1)x + \underbrace{O\left(\tfrac{x}{m}\right)}_{=O(m)\text{ because } m=\lfloor\sqrt{x}\rfloor \text{ and } x\geq 1.} + O(m).$$

Recall the Taylor expansion $\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \cdots$ for $|y|<1$.

$$= O(y), \text{ for } |y| \leq \tfrac{1}{2}.$$

We write $\log m = \log\left(\sqrt{x}\cdot\frac{m}{\sqrt{x}}\right) = \frac{1}{2}\log x + \log\left(\frac{\sqrt{x}-\{\sqrt{x}\}}{\sqrt{x}}\right)$

$= \frac{1}{2}\log x + \log\left(1 - \frac{\{\sqrt{x}\}}{\sqrt{x}}\right) = \frac{1}{2}\log x + O(1/\sqrt{x})$, for $x \geq 4$.

Thus
$$\sum_{n\leq x}\tau(n) = 2x\left(\tfrac{1}{2}\log x + O\left(\tfrac{1}{\sqrt{x}}\right)\right) + (2\gamma-1)x + O(\sqrt{x})$$
$$= x\log x + (2\gamma-1)x + O(\sqrt{x}), \text{ for } x\geq 4,$$
and in fact for $x \geq 1$. ☒

Theorem 3 (Abel's Summation) Let $a<b$ be real, $f\in C^1(a,b)$, $(a_n)\subset\mathbb{R}$ be a sequence of real numbers, and $A(x) := \sum_{n\leq x} a_n$ for $x\in\mathbb{R}$ (where the $\emptyset$ sum is defined as 0 and, as always, $n\in\mathbb{N}$). Then

$$\boxed{\sum_{a<n\leq b} a_n f(n) = A(b)f(b) - A(a)f(a) - \int_a^b A(t)f'(t)\,dt.}$$

Proof. Again by Titchmarsh's trick (additivity) – note that if the identity holds for $a<b$ and $b<c$, then (by summing) it also holds for $a<c$. Hence we may assume, as in the proof of Thm. 2, that $m \le a < b \le m+1$, $m \in \mathbb{Z}$. For such interval $[a,b]$ we can compute the $\int$ on the RHS by ~~integration by parts~~ primitive function:

$$\int_a^b A(t)\, f'(t)\,dt = \int_a^b A(m)\, f'(t)\,dt = A(m)\,(f(b)-f(a)).$$

Then the RHS of the identity becomes

$$A(b)f(b) - \underbrace{A(a)}_{=A(m)} f(a) - A(m)(f(b)-f(a)) = (A(b)-A(m))\,f(b) = \begin{cases} 0 & \ldots\ b<m+1, \\ a_{m+1} f(m+1) & \ldots\ b=m+1, \end{cases}$$

which agrees with the LHS $\sum_{a<n\le b} a_n f(n)$. ☒

We show two applications of Abel's summation.

Proposition For every real $x \ge 2$,

$$\sum_{p\le x} \frac{\log p}{p} = \log x + O(1) \Rightarrow \sum_{p\le x} \frac{1}{p} = \log(\log x) + c + O(1/\log x)$$

where c is a constant (and p denotes prime numbers).

Proof. We use Abel's summation for $a_n = \begin{cases} \frac{\log p}{p} & \ldots\ n=p, \\ 0 & \ldots\ n\ne p \end{cases}$, $[a,b]=[2-\varepsilon, x]$, $f(t)=\frac{1}{\log t}$. $\varepsilon>0$ small, $\varepsilon \to 0^+$. Then we have

$$\sum_{p\le x}\frac{1}{p}=\sum_{p\le x}\frac{\log p}{p}\cdot\frac{1}{\log p}=\sum_{a<n\le b}a_n f(n)=\frac{A(x)}{\log x}-\underbrace{\frac{\overbrace{A(2-\varepsilon)}^{=0}}{\log(2-\varepsilon)}}_{=0}-\int_2^x A(t)f'(t)\,dt.$$

By the assumption, $A(t)=\sum_{n\le t}a_n=\sum_{p\le t}\frac{\log p}{p}=\log t+R(t)$, $R(t)=O(1)$. Therefore:

$$=1+\frac{R(x)}{\log x}+\int_2^x\frac{dt}{t\log t}+\int_2^x\frac{R(t)\,dt}{t\log^2 t}=$$

$$\frac{R(x)}{\log x}=O(1/\log x)$$

$$=\log(\log x)+\Big(1-\log\log 2+\underbrace{\int_2^{+\infty}\frac{R(t)\,dt}{t\log^2 t}}_{\text{Converges}}\Big)+\Big(\frac{R(x)}{\log x}-\int_x^{+\infty}\frac{R(t)\,dt}{\underbrace{t\log^2 t}_{O\left(\frac{1}{t\log^2 t}\right)}}\Big)=$$

$\left[\int\frac{1}{t\log t}=\log\log t\ (+c)\right]$

$$=\boxed{\log\log x+c+O\left(\frac{1}{\log x}\right).}$$

This is because $\int\frac{1}{t\log^2 t}=-\frac{1}{\log t}\ (+c)$ $\square$

Unfortunately, I do not have time to show you the proof of the ~~a~~ hypothesis of the $\Rightarrow$, the so called

$$\boxed{\text{1st Mertens formula: } \forall x\ge 2,\ \sum_{p\le x}\frac{\log p}{p}=\log x+O(1)}$$

the conclusion, which we have deduced by ~~the~~ Abel's summation, is the so called

$$\boxed{\text{2nd Mertens formula: } \forall x\ge 2,\ \sum_{p\le x}\frac{1}{p}=\log\log x+c+O\left(\frac{1}{\log x}\right)}$$

We continue with the 2nd application of Abel's summation next time.