(L2) We continue in the proof of the FTAlg. (1)

(S1) $p\in\mathbb{C}[z]$ is non-constant and $|p|$ attains minimum at $z=0$. We show that $|p(0)|>0$, i.e. 0 is not a root, leads to a contradiction. So let

$$p(z) = a_0 + a_k z^k + a_{k+1} z^{k+1} + \dots + a_n z^n$$
$$=: a_0 + a_k z^k + q(z),$$

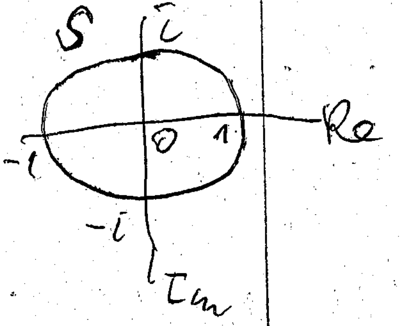
where $n\in\mathbb{N}$, $k\in\mathbb{N}$, $a_0, a_k \neq 0$, $q\in\mathbb{C}[z]$ and $q(z)=o(z^k)$ for $z\to 0$.

Using the assumption we take an $\alpha\in\mathbb{C}$ s.t. $\boxed{\alpha^k = -a_0/a_k}$. By $\supset$ we can take a $\delta\in(0,1)$ s.t. $|q(\delta\alpha)|\leq \frac{1}{2}\delta^k|a_0|$.

Then
$$|p(\delta\alpha)| = |a_0 - a_0\delta^k + q(\delta\alpha)|$$
$$\leq |a_0|(1-\delta^k) + |q(\delta\alpha)|$$
$$\leq |a_0|(1-\delta^k/2) < |a_0| = |p(0)|,$$
contradicting the minimality of $|p(0)|$. Thus $|p(0)|=0$ and 0 is a root of p. $\boxtimes$

(S2) We ~~want to~~ prove: $\forall k\in\mathbb{N}, \alpha\in\mathbb{C}\ \exists \beta\in\mathbb{C}: \beta^k=\alpha$. The case $k=2$, i.e. every complex number has a square root) is an exercise for you.

Also, every $\alpha \in [0,+\infty)$ has a k-th root for every $k \in \mathbb{N}$ (via supremum). Hence we can restrict to odd $k \in \mathbb{N}$ (any $k \in \mathbb{N}$ can be written as $k = 2^l k'$, $l \in \mathbb{N}_0$, k' odd) and $|\alpha| = 1$ (we replace α with $\frac{\alpha}{|\alpha|}$ (of course $\alpha \neq 0$)). Thus, if $\underline{S = \{z \in \mathbb{C} \mid |z| = 1\}}$



is the unit complex circle, and $\underline{k \in \mathbb{N} \text{ is odd}}$, we need to prove that the map

$$f(z) = z^k : S \to S \quad \text{is onto.}$$

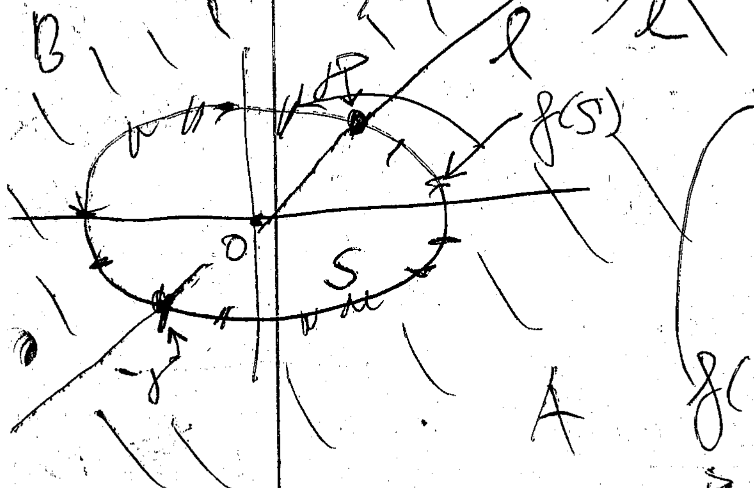
an open partition of X

$X \subset \mathbb{C}$ is <u>disconnected</u>: $\exists$ open sets $A, B \subset \mathbb{C}$ s.t. $X = (X \cap A) \cup (X \cap B)$ is a partition of X (both (...) are $\neq \emptyset$ and are disjoint). We say that X is <u>connected</u> iff it is not disconnected.

$\boxed{\text{Proposition}}$ $F : \mathbb{C} \to \mathbb{C}$ is continuous, $X \subset \mathbb{C}$ is connected $\Rightarrow$ $F(X) \subset \mathbb{C}$ is connected too.

<u>Proof</u> exercise for you.

Suppose, for contradiction, that $f(S) \subsetneq S$: $\exists \gamma \in S \setminus f(S)$. Since $f(z) = z^2$ and k is odd, also $-\gamma \in S \setminus f(S)$. Let A, B be the two open half-planes determined by the line going through γ and $-\gamma$:



the open partition ~~then $S = (S \cap A) \cup (S \cap B)$~~ shows that $f(S)$ is $f(S) = (f(S) \cap A) \cup (f(S) \cap B)$

(*) disconnected,* contrary to the Proposition.

Thus $f(S) = S$ and $\forall \alpha \in S\ \exists \beta \in S : \beta^2 = \alpha$. $\boxtimes$

Few comments on the two examples of the last lecture.

(*) But why is S connected? - an exercise

(1) Covering congruences: as we have seen, it is not possible to partition $\mathbb{N}$ in ≥ 2 APs with distinct common differences. But P. Erdős observed in 1950 that it is possible to express $\mathbb{N}$ as a union of such APs: $\forall n \in \mathbb{Z}$: $n \equiv 0\ (2)$ or $n \equiv 0\ (3)$ or $n \equiv 1\ (4)$ or $n \equiv 3\ (8)$ or $n \equiv 7\ (12)$ or $n \equiv 23\ (24)$.

(2) the theorem of Bob Hough (Ann. of Math., 2015):

Every system of covering congruences (with distinct moduli) has minimum modulus $< 10^{16}$.) P. Balister, ... 2 Lovest.

this answers in negative a question of P. Erdős

(3) A bijective proof for the Euler odd-parts $\times$ distinct parts identity can be given by using the representations of $n \in \mathbb{N}$ in the form $n = 2^{k} l$ where $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$ is odd and in ~~the~~ the form $n = 2^{m_1} + 2^{m_2} + \dots + 2^{m_k}$ where $0 \le m_1 < m_2 < \dots < m_k$ are integers (binary expansions) - exercise for you. Another combinatorial proof of that identity can be given by the inclusion exclusion principle.

Some real analysis tools ~~for~~ in A.N.T 5 theorems!

$\sum$s and $\int$s - we consider the Riemann $\int$s.

Theorem 1 $a, b \in \mathbb{Z}$, $a < b$, $f: [a,b] \to \mathbb{R}$ is monotonic $\Rightarrow \exists\, \theta \in [0,1]$ s.t.

$$\sum_{a < n \le b} f(n) = \int_a^b f + \theta \cdot (f(b) - f(a)).$$

$\theta = \theta(f, a, b)$, not $\theta = \theta(a,b)$! - exercise

[recall the theorem from Math. an. course that every monot. function is R.-integrable]

<u>Proof.</u> We take for the partition $D=(a,a+1,\dots,b)$ $\to I_1,\dots,I_k$; $k=b-a$ of $[a,b]$ the lower and upper Darboux sum of D and f:

$$s(f,D)=\sum_{i=1}^{k}\inf_{I_i} f(t)\cdot \underbrace{|I_i|}_{=1} \le \int_a^b f \le S(f,D)=\sum_{i=1}^{k}\sup_{I_i} f(t)\cdot \underbrace{|I_i|}_{=1}$$

$f:\searrow$ $\quad f(a+1)+f(a+2)+\dots+f(b)\le\int_a^b f\le f(a)+f(a+1)+\dots+f(b-1)$

$f:\nearrow$ $\quad f(a)+f(a+1)+\dots+f(b-1)\le$ — " — $\le f(a+1)+f(a+2)+\dots+f(b)$

Hence: $\searrow$: $0\le\int_a^b f-(f(a+1)+\dots+f(b))\le f(a)-f(b)$

for f $\nearrow$: $f(a)-f(b)\le$ — " — ≤ 0, and indeed

$$\sum_{a<n\le b} f(n)=\int_a^b f+\theta\cdot(f(a)-f(b)),\quad 0\le\theta\le 1.$$

$\boxtimes$

For example: Let $n\in\mathbb{N}$ $\quad \log(n!)=\log 2+\log 3+\dots+\log n=$

$$=\int_1^n \log x+\theta\cdot(\log 1-\log n)=[x\log x-x]_1^n+$$

$+\theta\cdot\log n=n\log n-n+1+\theta\cdot\log n$, $\theta\in[0,1]$, and

$$\boxed{n!=\frac{e}{n^{\theta}}\left(\frac{n}{e}\right)^n,\ 0\le\theta\le 1}$$

Another example: $H_n = \sum_1^n\frac{1}{j}=1+$

$$+\sum_{j=2}^{n}\frac{1}{j}=\int_1^n \frac{1}{x}+\theta\cdot\left(\frac{1}{n}-1\right)=$$

$$H_n=\boxed{\log n+1-\theta\left(1-\frac{1}{n}\right),\ 0\le\theta\le 1}$$

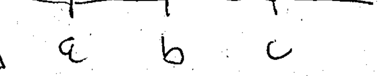
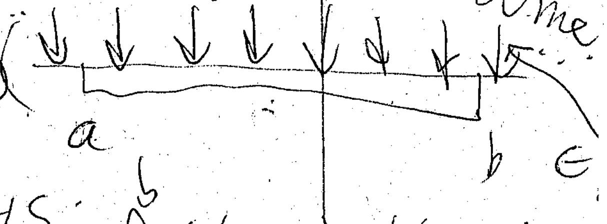
(n-th harmonic number)

Theorem 2 (Euler-MacLaurin f. for 1^{st} derivatives)

$a,b\in\mathbb{R}$, $a<b$, $f\in C^1(a,b)$. Then

$$\sum_{a<n\le b} f(n) = \int_a^b f + \int_a^b (t-\lfloor t\rfloor)\, f'(t)\,dt + (a-\lfloor a\rfloor)\, f(a) -$$

$$(*) \qquad - (b-\lfloor b\rfloor)\, f(b). \quad \Big[\text{Here } \lfloor x\rfloor = \max_{n\in\mathbb{Z},\, n\le x} n.\Big]$$

Proof (E.C. Titchmarsh, the theory of the Riemann zeta-function). The identity $(*)$ is additive: if true for $[a,b]$, $[b,c]$, then true also for $[a,c]$ . Thus it suffices to assume that $m\le a<b\le m+1$, $m\in\mathbb{Z}$ 

$\int$ by parts of the 2^{nd} term on the RHS: $\int_a^b (t-m)\, f'(t)\,dt =$

$$= \Big[(t-m) f(t)\Big]_a^b - \int_a^b f$$

$$= (b-m)\, f(b) - \int_a^b f. \text{ The RHS}$$

becomes ~~[illegible]~~ $-(a-m)\, f(a)$ (as $\lfloor a\rfloor = m$) ~~just~~

$(\lfloor b\rfloor - m)\, f(b) =$ 0 … if $b<m+1$; $f(b)$ … if $b=m+1$, which agree with the LHS ⊠

[In a monograph on ANT a proof takes $1\frac{1}{2}$ page.]

For example, what does it give for $f(x) = \frac{1}{x}$?