Lecture 1

M. Klazar

Oct 6, 2023

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, \dots\}$, $\mathbb{Z}$ be the integers, $\mathbb{Q}$ be the fractions and $\mathbb{R}$ be the real numbers. For $m, n \in \mathbb{Z}$ we write $(m, n) = 1$ to say that m and n are coprime, their largest common divisor is 1. Every number $\alpha \in \mathbb{R}$ decomposes uniquely as the sum

$$\alpha = \lfloor \alpha \rfloor + \{\alpha\}$$

of its *(lower) integer part* $\lfloor \alpha \rfloor \in \mathbb{Z}$ and its *fractional part* $\{\alpha\} \in [0, 1)$.

Theorem (P. Dirichlet, 1842) *For every $\alpha \in \mathbb{R}$ and every $Q \in \mathbb{N}$ with $Q \geq 2$ there exist $p, q \in \mathbb{Z}$ such that $1 \leq q < Q$ and*

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{Qq}\ .$$

To prove it we consider Q numbers $\{n\alpha\} \in [0, 1)$ for $n = 0, 1, \dots, Q-1$. We can think of them as points lying on a circle with circumference 1. Two of them have arc distance $\leq 1/Q$, which means that

$$|m\alpha - r - (n\alpha - s)| \leq 1/Q$$

for some $m, n, r, s \in \mathbb{Z}$ with $0 \leq n < m < Q$. We set $p := r - s$, $q := m - n$, divide the inequality by q and get that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{Qq} \ \text{ and } \ 1 \leq q < Q\ .$$

QED

Corollary *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there exist infinitely many distinct fractions p/q such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}\ .$$

We prove it by constructing infinitely many fractions p_n/q_n for $n \in \mathbb{N}$ such that for each the displayed inequality holds and $|\alpha - p_1/q_1| > |\alpha - p_2/q_2| >$

$\cdots > 0$. We begin with $p_1 := \lfloor\alpha\rfloor$ and $q_1 := 1$. If $p_1/q_1,\ \ldots,\ p_n/q_n$ are already constructed, we take any $Q \in \mathbb{N}$ such that $|\alpha - p_n/q_n| > 1/Q$ (this is possible, α is irrational and always $|\cdots| > 0$) and use Dirichlet's theorem. We get a fraction p/q such that $1 \le q < Q$ and $|\alpha - p/q| < 1/Qq < 1/q^2$. Also, $|\alpha - p/q| < 1/Q < |\alpha - p_n/q_n|$. Thus we can set $p_{n+1} := p$ and $q_{n+1} := q$.
QED

To obtain an ultimate strengthening of this corollary, the theorem of Hurwitz, we need so called Farey fractions and their properties. For every $n \in \mathbb{N}$ we consider the ordered list

$$F_n := \left(\tfrac{0}{1} = \tfrac{p_1}{q_1} < \tfrac{p_2}{q_2} < \cdots < \tfrac{p_m}{q_m} = \tfrac{1}{1}\right)$$

of all $m = m(n)$ fractions $p/q \in [0,1]$ such that $0 < q \le n$ and $(p,q) = 1$. These are the *Farey fractions (of order n)*. For example,

$$F_5 = \left(\tfrac{0}{1} < \tfrac{1}{5} < \tfrac{1}{4} < \tfrac{1}{3} < \tfrac{2}{5} < \tfrac{1}{2} < \tfrac{3}{5} < \tfrac{2}{3} < \tfrac{3}{4} < \tfrac{4}{5} < \tfrac{1}{1}\right).$$

Theorem (Ch. Haros, 1802) *If $\frac{a}{b} < \frac{c}{d}$ are two consecutive fractions in the list F_n then*

$$\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}\,,\ \textit{that is, } bc - ad = 1\,.$$

In the proof we show that the Diophantine equation

$$bx - ay = 1$$

is solved by $x = c, y = d$. Since $(a,b) = 1$, there is at least one solution $x_0, y_0 \in \mathbb{Z}$. This follows from the fact that in the ring $\mathbb{Z}$ every ideal, such as $\{ua + vb \mid u, v \in \mathbb{Z}\}$, is principal, is generated by a single element; it follows from the division with remainder. Thus $bx_0 - ay_0 = 1$ and we see that $x = x_0 - ra$ and $y = y_0 - rb$ is also a solution for any $r \in \mathbb{Z}$. It follows that there is a solution $x_1, y_1 \in \mathbb{Z}$ such that

$$n - b < y_1 \le n\,.$$

From $bx_1 - ay_1 = 1$ we get the equality

$$\frac{x_1}{y_1} = \frac{1}{by_1} + \frac{a}{b}\,.$$

We show that x_1/y_1 is in the list F_n: from the above we see that $1 \le y_1 \le n$ and that $(x_1, y_1) = 1$, and from $bx_1 - ay_1 = 1$ and $0 < a < b$ it follows that $0 < x_1 \le y_1$. From $x_1/y_1 > a/b$ we thus get that $x_1/y_1 \ge c/d$.

We assume that $x_1/y_1 > c/d$ and deduce a contradiction. By adding the trivial inequalities

$$\frac{x_1}{y_1} - \frac{c}{d} \ge \frac{1}{dy_1} \ \text{ and } \ \frac{c}{d} - \frac{a}{b} \ge \frac{1}{bd}$$

we get

$$\frac{1}{by_1} = \frac{x_1}{y_1} - \frac{a}{b} \geq \frac{1}{dy_1} + \frac{1}{bd} = \frac{b+y_1}{bdy_1} \text{ and } d \geq b + y_1 .$$

But above we see that $b+y_1 > n$ and have the contradiction $d > n$, as $c/d \in F_n$.

Thus $x_1/y_1 = c/d$. These are fractions in lowest terms and $x_1 = c$, $y_1 = d$ is a solution of $bx - ay = 1$.

QED

The distance between two consecutive fractions $\frac{a}{b} < \frac{c}{d}$ in F_n is therefore minimum possible (for two distinct fractions). Clearly, $0 < \frac{c}{d} - \frac{a}{b} \leq \frac{1}{n}$. It is interesting that their *mediant* $\frac{a+c}{b+d}$, which need not be a Farey fraction, lies in the minimum distance to each:

$$(a+c)b - (b+d)a = cb - da = 1 \text{ and } (b+d)c - (a+c)d = bc - ad = 1 .$$

If $\frac{a}{b} < \frac{c}{d} < \frac{e}{f}$ are three consecutive fractions in F_n then, again interestingly,

$$\frac{a+e}{b+f} = \frac{c}{d} ,$$

the middle fraction is the mediant of the outer two — prove it as an exercise.

Theorem (A. Hurwitz, 1891) *For every $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ there exist infinitely many distinct fractions p/q such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5} \cdot q^2} .$$

On the other hand, for every real $c > \sqrt{5}$, the inequality

$$\left|\frac{\sqrt{5}-1}{2} - \frac{p}{q}\right| < \frac{1}{cq^2}$$

has only finitely many solutions $p/q \in \mathbb{Q}$.

In the proof of the first claim we make use of Farey fractions. We may assume (by replacing α with $\{\alpha\}$) that $0 < \alpha < 1$. Like in the proof of the above corollary, we construct a sequence of fractions $p_n/q_n \in [0,1]$ for $n \in \mathbb{N}$ such that for each the first displayed inequality holds and $|\alpha - p_1/q_1| > |\alpha - p_2/q_2| > \cdots > 0$. We claim that $\frac{p_1}{q_1}$ can be always one of the three fractions $\frac{0}{1}$, $\frac{1}{2}$ and $\frac{1}{1}$. It follows from the fact that the sum of lengths of the intervals $[0, \frac{1}{\sqrt{5}}]$ and $[\frac{1}{2} - \frac{1}{4\sqrt{5}}, \frac{1}{2}]$ is larger than the length of $[0, \frac{1}{2}]$: $\frac{1}{\sqrt{5}} + \frac{1}{4\sqrt{5}} > \frac{1}{2}$ as $\frac{5}{16} > \frac{1}{4}$. If p_1/q_1, $\dots$, p_n/q_n are constructed, we take $m \in \mathbb{N}$ so large that $|\alpha - p_n/q_n| > 1/m > 0$ (recall that α is irrational), take two consecutive fractions in the list F_m such that

$$\frac{a}{b} < \alpha < \frac{c}{d}$$

and show that one of these two Farey fractions or their mediant $\frac{e}{f} := \frac{a+c}{b+d}$ has the required properties.
QED

Remarks The first theorem in this lecture appeared in [1], the second one in [2] and the third one in [3]. It goes without saying that their proofs here may differ from the original ones.

References

[1] L. P. G. Dirichlet, Verallgemeinerung eines Satzes aus der Lehre von den Kettenbrüchen nebst einigen Anwendungen auf die Theorie der Zahlen, *S. B. Preuss. Akad. Wiss.* (1842), 93–95

[2] C. Haros, Tables pour évaluer une fraction ordinaire avec autand de decimals qu'on voudra; et pour trouver la fraction ordinaire la plus simple, et dui a approche sensiblement d'une fraction décimale, *Journal de l'École Polytechnique* **4** (1802), 364–368

[3] A. Hurwitz, Über die angenäherte Darstellung der Irrationalen durch rationale Brüche, *Math. Annalen* **39** (1891), 279–284

[4] P. Pavlíková, O Fareyových zlomcích, *Pokroky matematiky, fyziky a astronomie* **55** (2010), 97–110

[5] V. M. Schmidt, *Diophantine Approximation*, Springer-Verlag, Berlin 1980

[6] V. Šmidt, *Diofantovy približenija*, Mir, Moskva 1983