MATHEMATICAL STRUCTURES (NMAI064)
summer term 2024/25
lecturer: Martin Klazar

LECTURE 11 (April 30, 2025)

TOPOLOGY: WAYS OF INTRODUCING IT, EXAMPLES, AND CONTINUOUS MAPS

(based on the lecture notes of A. Pultr, Chapter V.1–V.3)

• *Review of metric spaces.* One may approach the concept of space in several ways. The students usually encounter first *the metric space.* A set of points X is endowed with a non-negative real function ρ on the Cartesian product $X \times X$ (the metric or distance function) such that

$\rho(x, y) = 0$ if and only if $x = y$.

$\rho(x, y) = \rho(y, x)$ (symmetry).

$\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ (the triangle inequality).

A mapping $f\colon X \to Y$, written $f\colon (X, \rho) \to (Y, \sigma)$, between metric spaces is *continuous* if

$$\forall\, x \in X\ \forall \varepsilon > 0\ \exists\, \delta > 0 :\ \rho(x,\, y) < \delta \Rightarrow \sigma(f(x),\, f(y)) < \varepsilon\ .$$

Exercise 1 *Show that one can deduce the nonnegativity of ρ from the three axioms.*

Exercise 2 *Let $X := \mathcal{R}(a, b)$ (the Riemann-integrable functions $f\colon [a, b] \to \mathbb{R}$) and*

$$\rho(f,\, g) := \int_a^b |f(t) - g(t)|\ \mathrm{d}t\ .$$

Decide if (X, ρ) is a metric space.

On the other hand, students learn very soon that for many purposes (for instance, for basic problems of mathematical analysis) a concretely chosen metric is not all that important — for instance, in the Euclidean plane we can take, instead of the geometrically intuitive distance

$$\rho((x_1,\, x_2),\, (y_1,\, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

a more convenient distance

$$\max(|x_1 - y_1|,\ |x_2 - y_2|)$$

and everything that concerns continuity remains valid.

• *Birth of neighborhoods.* In 1914, *Felix Hausdorff* introduced a continuity structure based on the notion of neighborhood. It took hold (in several variants — later we will prefer the approach using the so called open sets). The intuition is very satisfactory: it models the concept of being surrounded as opposed to be "just on the border". For instance, a point a in the Euclidean plane for which $\rho(a, (0,0)) < 1$ is surrounded by the set $M = \{x \mid \rho(x, (0,0)) \leq 1\}$ while a point b such that $\rho(b, (0,0)) = 1$ is not, although it also belongs to the set M. The continuity is then defined by requiring that

> for every $x \in X$ and for every neighborhood V of $f(x)$ there is a neighborhood U of x such that $f[U] \subseteq V$.

Examples. 1. On the set of real numbers take an M for a neighborhood of x if there exist numbers a, b such that $a < x < b$ and $\{y \mid a < y < b\} \subseteq M$. Note that the mappings continuous in this sense coincide with those continuous in the standard metric ε, δ-definition.

2. The notion of neighborhood simplifies the situation. For instance, take the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$, define the neighborhoods of the $x \in \mathbb{R}$ as before, and for $+\infty$ (resp. ∞) take the M for which there is a number K with $\{x \mid x > K\} \subseteq M$ (resp. $\{x \mid x < K\} \subseteq M$). Then the formula

> for every neighborhood U of b there is a neighborhood V of a such that $f[V \setminus \{a\}] \subseteq U$

defines the limit of a function f in a as b for any a and b, finite or infinite.

• *Neighborhoods.* A *topology on a set X is defined by neighborhoods* if every element $x \in X$ is assigned a *non-empty* set $\mathcal{U}(x) \subseteq \mathcal{P}(X)$ such that

(nb1) For every $U \in \mathcal{U}(x)$ one has that $x \in U$.

(nb2) If $U, V \in \mathcal{U}(x)$ then $U \cap V \in \mathcal{U}(x)$.

(nb3) If $U \in \mathcal{U}(x)$ and $U \subseteq V \subseteq X$ then $V \in \mathcal{U}(x)$.

(nb4) For every $U \in \mathcal{U}(x)$ there exists a $W \in \mathcal{U}(x)$ with $W \subseteq U$ and such that for every $y \in W$ one has that $U \in \mathcal{U}(y)$.

Exercise 3 *If the conditions* (nb1)–(nb4) *hold, then we also have a formally stronger*

(nb5) *For every $U \in \mathcal{U}(x)$ there exists a $W \in \mathcal{U}(x)$ with $W \subseteq U$ and such that for every $y \in W$ one has that $W \in \mathcal{U}(y)$.*

• *Open sets.* Now we will present another approach (shortly we will see that it is equivalent with the previous one). We say that we have a *topology on a set X defined by open sets* if there is given a set $\tau \subseteq \mathcal{P}(X)$ such that

(op1) $\emptyset, X \in \tau$.

(op2) If $U, V \in \tau$ then $U \cap V \in \tau$.

(op3) If $U_i \in \tau$ for every $i \in I$ then $\bigcup_{i \in I} U_i \in \tau$.

The elements $U \in \tau$ are called *open sets*. The concept of an open set is not quite as intuitive as that of a neighborhood, but it is much easier to work with. Note, too, that there is no requirement similar to the not very transparent (nb4).

The two approaches presented (and similarly further ones to be mentioned later) are equivalent in the following sense: if we complete the system by suitable definition of the other notion (as a derived one), we will obtain the same, that is, the same system of (here two) concepts. Let us discuss it in some more detail.

If we have a topology $\mathcal{U}$ on X in the sense of neighborhoods, define $\tau \subseteq \mathcal{P}(X)$ by setting

$$U \in \tau \text{ iff } U \in \mathcal{U}(x) \text{ for all } x \in U \tag{*}$$

(that is, U is open if it is a neighborhood of each of its points). The reader readily checks that such τ satisfies the requirements (op1) through (op3).

If we have a topology $\tau \subseteq \mathcal{P}(X)$ in the sense of open sets, define neighborhoods $\mathcal{U}$ as follows:

$$U \in \mathcal{U}(x) \text{ if there exists a } V \in \tau \text{ such that } x \in V \subseteq U\,. \tag{**}$$

Again, it is easy to see that such $\mathcal{U}$ satisfies conditions (nb1) through (nb4).

Start with $\mathcal{U}$ and construct τ by (*); now, take this τ and define $\mathcal{U}'$ using (**). If $U \in \mathcal{U}(x)$ choose V by Exercise 3; then $V \in \tau$ and hence $U \in \mathcal{U}'(x)$. If $U \in \mathcal{U}'(x)$ take the V from (**). This is in $\mathcal{U}(x)$ and hence also $U \in \mathcal{U}(x)$. Thus, $\mathcal{U}' = \mathcal{U}$.

Start with a given τ and define $\mathcal{U}$ by (**), and then define τ' from $\mathcal{U}$ by (*). If $U \in \tau$ then $U \in \mathcal{U}(x)$ for every $x \in U$ (we can take U itself for the V) and hence $U \in \tau'$. If $U \in \tau'$ choose for each $x \in U$ a set $V_x \in \tau$ so that $x \in V_x \subseteq U$ and obtain $U = \bigcup_{x \in U} V_x \in \tau$ by (op3). Hence $\tau = \tau'$

• *Closed sets.* Let a topology on X be given as a system of open sets. A subset $A \subseteq X$ is said to be *closed* if $X \setminus A$ is open. From the De Morgan laws we immediately obtain that

> a union of finitely many and an intersection of arbitrarily many closed sets is closed.

We can, of course, start with a system that has this property and define open sets as complements of the closed ones.

• *Closure.* Let (X, τ) be a topology. For $M \subseteq X$ define the closure of M by setting

$$\overline{M} := \bigcap \{A \mid M \subseteq A \subseteq X \text{ and } A \text{ is closed}\} .$$

Since the intersection of an arbitrary system of closed sets is closed, we see that $\overline{M}$ is the inclusion-wise least closed set containing M.

Exercise 4 *Prove the following properties of the closure operation.*

1. $M \subseteq \overline{M}$ *and* $\overline{\emptyset} = \emptyset$.
2. $M \subseteq N \Rightarrow \overline{M} \subseteq \overline{N}$.
3. $\overline{M \cup N} = \overline{M} \cup \overline{N}$.
4. $\overline{\overline{M}} = \overline{M}$.

If we wish to define the closure out of the neighborhoods, we will find useful the following formula:

$$\overline{M} = \{x \in X \mid \forall \text{ neighborhood } U \text{ of } x \text{ one has that } U \cap M \neq \emptyset\} .$$

Exercise 5 *Prove it.*

A topology can be defined by starting with the closure as well. We take for the basic notion the closure as a mapping u from $\mathcal{P}(X)$ to $\mathcal{P}(X)$, $M \mapsto \overline{M}$, satisfying the formulas from Exercise 4. Then we can define open sets U as those for which $u(X \setminus U) = X \setminus U$, and $M \subseteq X$ is a neighborhood of x if $x \notin u(X \setminus M)$. It is a useful exercise to check the equivalence similarly as in (*) and (**) above.

And one more definition: the *interior of* $M \subseteq X$ is the largest open set contained in M. The operation of interior has the properties (dually) analogous to those of closure (precisely which?), and it can be, again, taken as the basic notion for the topology.

Exercise 6 *Describe in detail the dual properties of interiors.*

Summary and definition. A topological space is a set together with a topology defined by any of the ways described above. It does not matter with which of the notions we start (neighborhoods, open or closed sets, closure, interior); we work with all of them anyway. In these lectures, because of the technical simplicity, we will usually start with the open sets.

• *Examples.* We first consider *metric spaces.* In a metric space (X, ρ), we define for $x \in X$ and $\varepsilon > 0$ the set

$$\Omega(x,\, \varepsilon) := \{y \in X \mid \rho(x,\, y) < \varepsilon\}$$

(the open ball with center x and radius ε). A neighborhood of an $x \in X$ is any $M \subseteq X$ such that, for a sufficiently small ε, $\Omega(x, \varepsilon) \subseteq M$. An open set is one that is in this sense a neighborhood of each of its points (of course), that is, such a U that for every $x \in U$ there is an ε such that $\Omega(x, \varepsilon) \subseteq M$. For a point $x \in X$ and a subset $A \subseteq X$ define $\rho(x, A) := \inf(\{\rho(x, a) \mid a \in A\})$. Set

$$\overline{A} := \{x \in X \mid \rho(x,\, A) = 0\}$$

and declare A to be closed if $\overline{A} = A$. Check that the relations between thus defined notions agree with the above definitions. A topological space obtained in this way from a metric one is said to be *metrizable.*

Discrete space. On a set X take for τ the whole of $\mathcal{P}(X)$. Thus, all the $M \subseteq X$ are both open and closed, every set containing a point x

is its neighborhood, $\overline{M} = M$ for any $M \subseteq X$. In this case we speak of the *discrete topology* on X.

Indiscrete space. This is the opposite extreme: take for open (and closed) sets the $\emptyset$ and X only. Then each point has one and only neighborhood, namely the whole of X, and the closure of any non-void set is the whole space as well.

Cofinite topology. This is only a little less primitive case: the open sets are $\emptyset$ and the complements of finite sets. The closed sets are then precisely the finite ones and the whole space, and the closure of an infinite set is the whole space.

Exercise 7 *Check that cofinite topology is indeed a topology.*

Alexandroff (quasidiscrete) topology. Let $(X, \leq)$ be a preordered set (the following definition is usually applied for posets, but a pre-order suffices). Recall the previous notation $\uparrow M$ and $\downarrow M$. In the Alexandroff topology, the open sets are all the increasing sets (that is, the $U \subseteq X$ such that $\uparrow U = U$). The closed sets are then all the decreasing ones (that is, the $U \subseteq X$ with $\downarrow U = U$, and the closure is given by $\overline{M} = \downarrow M$. Note that in this topology

(qd) *all* the intersections of open sets are open, *all* unions of closed sets are closed, and $\overline{\bigcup_{i \in J} M_i} = \bigcup_{i \in J} \overline{M_i}$ for any system of subsets.

This is why the Alexandroff spaces are often termed the *quasidiscrete* spaces. As a simple exercise prove that for every space satisfying condition (qd) there is a preorder such that the topology is given as described (define $x \leq y$ by $\overline{\{y\}} \subseteq \overline{\{x\}}$).

Scott topology. Again, start with a partially ordered set $(X, \leq)$. Now, declare a subset $U \subseteq X$ for open

(Sc) if it is increasing and if for every directed set D, $\sup D \in U$ implies that $U \cap D \neq \emptyset$.

This topology plays an important role in theoretical computer science.

• *Bases and subbases of a topology.* A *basis* of a topology τ (presented as the system of open sets) is any subset $\mathcal{B} \subseteq \tau$ such that

$$\forall U \in \tau : \ U = \bigcup \{B \in \mathcal{B} \mid B \subseteq U\} .$$

Note that a basis can be much simpler and more transparent than the whole of the topology: thus for instance we can take just the open intervals as a basis of the topology of the real line (and we can do just with the open intervals (a, b) with rational a, b); or, a basis of the topology of the plane can be reduced just to the open squares.

A *subbasis* of a topology τ is any $\mathcal{S} \subseteq \tau$ such that the set of all finite intersections of the elements of $\mathcal{S}$ is a basis of τ. The subbases can be, of course, again much simpler than bases. For the topology of the real line, for instance, we can now do just with the subbasis $\{(-\infty,\, a),\, (a,\, +\infty) \mid a \in \mathbb{Q}\}$.

Exercise 8 *Every subset $\mathcal{S}$ of $\mathcal{P}(X)$ is a subbasis of a topology, namely of the smallest topology in which all the $U \in \mathcal{S}$ are open. This topology is then said to be* generated by the *$\mathcal{S}$.*

Proposition 9 (on bases) *A set system $\mathcal{B} \subseteq \mathcal{P}(X)$ is a basis of a topology (X, τ) if and only if the following two conditions hold.*

1. *$X = \bigcup \mathcal{B}$.*
2. *If $A, B \in \mathcal{B}$ then $A \cap B = \bigcup\{C \in \mathcal{B} \mid C \subseteq A \cap B\}$.*

Exercise 10 *Prove this proposition.*

• *Two more examples.* The *interval topology.* Let $(X, \leq)$ be a linearly ordered set such that *there is in X neither minimum nor maximum element.* The set of all the intervals $(a, b) := \{x \in X \mid a < x < b\}$, $a, b \in X$, constitutes a basis of a topology; this topology is called the *interval topology* on $(X, \leq)$.

Exercise 11 *Check that the intervals (a, b) indeed form a basis of a topology.*

Sorgenfrey line. And one more, a little bizarre topology on the set of real numbers (it can be used for the so called semicontinuity, but also in theory to illustrate some weird phenomena). The *Sorgenfrey topology* is generated by the half-closed intervals; more precisely, it has the basis $\mathcal{B} := \{[a, b) \mid a, b \in \mathbb{R}\}$ where $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$.

Exercise 12 *Prove that $\mathcal{B}$ is indeed a basis of a topology on $\mathbb{R}$.*

Proposition 13 (on Sorgenfrey line) *In contrast to the standard Euclidean topology on $\mathbb{R}$, the Sorgenfrey topology on $\mathbb{R}$ has no finite or countable basis.*

Proof. Let $U_i \subseteq \mathbb{R}$, $i \in I$, be a system of open sets in the Sorgenfrey topology that forms its basis. Thus each U_i is a union of several intervals of the form $[a, b)$ and each interval $[a, b)$, $a < b$, is the union

$$[a,\, b) = \textstyle\bigcup_{i \in J} U_i$$

for some $J \subseteq I$. The key observation is that then there exists a real number c with $c > a$ (and, in fact, $c \le b$) and an index $j \in J$ such that $[a, c) \subseteq U_j$. Therefore we may define a map $f\colon \mathbb{R} \to I$ by $f(a) := j$, where $j \in J$ is that index for the interval $[a, b) = [a, a + 1)$. It is not possible that

$$f(a) = f(a') = j \in I$$

for some real numbers $a < a'$, because it would mean that for some real numbers $c > a$ and $c' > a'$ we have that

$$[a,\, c) \subseteq U_j \subseteq [a,\, a+1) \wedge [a',\, c') \subseteq U_j \subseteq [a',\, a'+1)$$

and $a \in [a', a' + 1)$, which is not the case. Thus f is injective and I is uncountable. □

• *Continuous mappings.* We say that a mapping $f\colon X \to Y$ is a *continuous mapping* from a topology (X, τ) to another topology (Y, σ), written also as $f\colon (X, \tau) \to (Y, \sigma)$, if for every $x \in X$ and every neighborhood V of $f(x)$ in the topology σ there exists a neighborhood U of x in the topology τ such that $f[U] \subseteq V$.

Exercise 14 *Prove that f is continuous if and only if*

$$\forall V \in \sigma:\ f^{-1}[V] \in \tau\ .$$

Prove this equivalence for closed sets.

Exercise 15 *Prove that the composition of two continuous maps between topological spaces is a continuous map.*

If X is endowed with the discrete topology or if Y is endowed with the indiscrete one, then every mapping $f\colon X \to Y$ is continuous. From the previous we also see the following.

Corollary 16 *Let $\mathcal{S}$ be a subbasis of a topology σ. Then $f\colon (X,\tau) \to (Y,\sigma)$ is continuous if and only if for every $U \in \mathcal{S}$ one has $f^{-1}[U] \in \tau$.*

Proposition 17 (a continuous map) *Let D be a subset of the interval $I = [0,1]$ (which we assume endowed with the topology induced by the standard metric) such that for any two numbers $a < b$ in I there exists a $d \in D$, such that $a < d < b$. Let U_d, $d \in D$, be open sets in a topological space X such that*

$$d < e \Rightarrow \overline{U_d} \subseteq U_e \ .$$

Then the mapping $f\colon X \to I$ defined by

$$f(x) := \inf(\{d \in D \mid x \in U_d\})$$

is continuous.

Proof. For $a \in (0,1)$ we have that

$$f(x) > a \iff x \notin \bigcap\{\overline{U_d} \mid a < d\} \iff x \in X \setminus \bigcap\{\overline{U_d} \mid a < d\} \ .$$

Also,

$$f(x) < a \iff x \in \bigcup\{U_d \mid a > d\} \ .$$

Thus $f^{-1}[(a,1]]$ and $f^{-1}[[0,a)]$ are open sets in X. Since

$$\{(a,1],\ [0,a) \mid a \in I\}$$

constitutes a subbasis of the space I, the statement follows. $\square$

We will need this proposition later on.

• *Homeomorphisms.* If for a continuous mapping $f\colon (X,\tau) \to (Y,\sigma)$ between topological spaces there exists an inverse mapping $g\colon (Y,\sigma) \to (X,\tau)$ that is also continuous, we say that f is a *homeomorphism* and that the spaces (X,τ) and (Y,σ) are *homeomorphic*.

Exercise 18 *Let $S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ be the unit circle in the plane and the map $f\colon [0,2\pi) \to S$ be given by*

$$f(t) = (\cos t,\ \sin t) \ .$$

Is f a homeomorphism between the Euclidean topological spaces $[0,2\pi)$ and S?

THANK YOU!

HOMEWORK: Exercises 2, 4, 10 and 18. Deadline is the end of the coming Monday. Please, send me your solutions by e-mail to klazar@kam.mff.cuni.cz. To get credits for the tutorial, you should solve (or at least send in attempted solutions of) at least half of the homework exercises.