MATHEMATICAL STRUCTURES (NMAI064)
summer term 2023/4
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LECTURE 11 (April 30, 2024). TOPOLOGY: STANDARD CONSTRUCTIONS AND SEPARATION AXIOMS
(based on the lecture notes of A. Pultr, Chapter V.4–V.5)

• *Subspaces.* Let (X,τ) be a topological space and let $Y \subseteq X$. We easily check that

$$\tau\,|\,Y := \{U \cap Y \mid U \in \tau\}$$

constitutes a topology on Y. This topology is referred to as the *topology of subspace*, or the *topology induced on the subset.*

Exercise 1 *Let $Y \subseteq X$ and (X,τ) be a topology. Prove that $(Y,\tau\,|\,Y)$ is a topological space. Give an example of sets $Z \subseteq Y \subseteq X$ such that Z is open in the space Y but not in the space X.*

For the embedding mapping $j\colon Y \to X$, $j(y) = y$, we have $Y \cap U = j^{-1}[U]$. Thus, $j\colon (Y,\tau\,|\,Y) \to (X,\tau)$ is a continuous map. Moreover, this topology is extreme in the sense that $\tau\,|\,Y$ is the least system of open sets necessary to make this map continuous. This has an important consequence.

Proposition 2 (on subspaces) *Let Y be a subspace of a space X, $j\colon Y \to X$ be the embedding mapping, and $g\colon Z \to Y$ be a map. If the map $jg\colon Z \to X$ is continuous, then g is continuous too.*

Exercise 3 *Prove this proposition.*

We observe that for any space (X,τ) and $Y \subseteq X$, in the space $(Y,\tau\,|\,Y)$ the closed sets are precisely the sets of the form $A \cap Y$ where A is closed in X, the closure of a set M is obtained from the original one as $\overline{M} \cap Y$, and the neighborhoods are the intersections of the original neighborhoods with Y.

The embedding of a subspace. Slightly more generally, if (X,τ) and (Y,θ) are topological spaces, if $j\colon Y \to X$ is a one-one mapping, and if

$$\theta := \{j^{-1}[U] \mid U \in \tau\}\ ,$$

we speak of the j as of an *embedding of a subspace.* Realize that this is precisely the case in which the restriction $(x \mapsto j(x)) : Y \to j[Y]$ of the mapping j is a homeomorphism.

• *Products.* Let us have a system (X_i, τ_i), $i \in J$, of topological spaces. On the Cartesian product $\prod_{i \in J} X_i$ we define a topology τ by the subbasis

$$\{p_j^{-1}[U] \mid j \in J, U \in \tau_j\} ,$$

where $p_j \colon \prod_{i \in J} X_i \to X_j$ are the standard projections $(x_i)_{i \in J} \mapsto x_j$. The topological space thus obtained is called the *product of the system* (X_i, τ_i)*,* $i \in J$, and if we wish to emphasize that we speak of this space and not just of its carrier $\prod_{i \in J} X_i$, we write

$$\prod_{i \in J} (X_i, \tau_i) .$$

For finite systems we write

$$(X_1, \tau_1) \times (X_2, \tau_2), \ X \times Y \times Z, \ X_1 \times \cdots \times X_n$$

etc.

Note that the subspaces and product are projectively generated. The topology is determined by the requirement that the preferred maps (in the first case the embedding, in the second one the projections) be continuous with the least possible systems of open sets. Similarly below, the factor (quotient) space and the sum will be injectively generated. Also note that for the finite systems of metric spaces we obtain the topology in agreement with the products of metric spaces as known from the course of mathematical analysis.

Theorem 4 (on products) *Let*

$$f_i \colon (Y, \theta) \to (X_i, \tau_i), \ i \in J ,$$

be a system of continuous mappings. Then there is precisely one continuous mapping

$$f \colon (Y, \theta) \to \prod_{i \in J} (X_i, \tau_i)$$

such that $p_i f = f_i$ *for all* $i \in J$*.*

Proof. The set-theoretic mapping f such that $p_i f = f_i$ for all $i \in J$ (namely, $f(y) = (f_i(y))_{i\in J}$) is continuous by Corollary 16 in the last lecture (the subbasis criterion): we have

$$f^{-1}[p_i^{-1}[U]] = f_i^{-1}[U]\,.$$

□

• *Factor (quotient) space.* Let (X, τ) be a topological space and let $q\colon X \to Y$ be a mapping onto (in particular we have in mind the situation where there is an equivalence E on X and q is the projection $(x \mapsto Ex)\colon X \to X/E$). We define a topology

$$\theta := \{U \subseteq Y \mid q^{-1}[U] \in \tau\}$$

on Y. This is, again, an extreme topology (this time the largest one) such that the particular mapping (here, $q\colon X \to Y$) is continuous. We speak of the *factor* or *quotient topology*, and of the *factor* or *quotient space*, or simply of the *quotient*. Analogically as Proposition 2 we (you) easily prove the next one.

Proposition 5 (on quotients) *Let Y be a quotient of a space X under the mapping $q\colon X \to Y$ and $g\colon Y \to Z$ be a map. If $gq\colon X \to Z$ is continuous, then also g is continuous.*

Exercise 6 *Prove this proposition.*

• *Sums (coproducts).* Let us have a system (X_i, τ_i), $i \in J$, of topological spaces. On the disjoint union

$$\coprod_{i\in J} X_i := \bigcup_{i\in J} X_i \times \{i\}$$

we define a topology τ by the basis

$$\{\iota_i[U] \mid i \in J,\, U \in \tau_i\}\ ,$$

where $\iota_j\colon X_j \to \coprod_{i\in J} X_i$ are the injections $x \mapsto (x, j)$. The obtained topological space is referred to as the *sum*, or *coproduct*, of the system (X_i, τ_i). If already the sets X_i are disjoint, we use for the carrier their union $\bigcup_{i\in J} X_i$.

Theorem 7 (on sums) *Let*

$$f_i\colon (X_i,\,\tau_i) \to (Y,\,\theta),\ \ i \in J\ ,$$

be a system of continuous mappings. Then there exists precisely one continuous mapping

$$f\colon \coprod_{i\in J}(X_i,\,\tau_i) \to (Y,\,\theta)$$

such that $f\iota_i = f_i$ *for all* $i \in J$.

Proof. Of course, it is the mapping defined by $f((x,i)) = f_i(x)$. □

And some more terminology: if τ and θ are topologies on the same set and if $\tau \subseteq \theta$, we say that τ is *weaker* (*coarser*) than θ, and that θ is *stronger* (*finer*) than τ.

• *Separation axioms* T_i. These are various conditions on a topology (X,τ) ensuring sufficient richness of the set system τ of open sets. Roughly speaking, with increasing index i these axioms define richer and richer topologies.

• T_0 *topologies.* We say that a space (X,τ) satisfies the *axiom* T_0 (or that it is a T_0*-space*) if

$$\forall\, x,\, y \in X,\, x \neq y\ \exists\, U \in \tau:\ (x \in U \not\ni y) \vee (y \in U \not\ni x)\ . \qquad (T_0)$$

Exercise 8 X *is a* T_0*-space if and only if*

$$\forall\, x,\, y \in X:\ \overline{\{x\}} = \overline{\{y\}} \Rightarrow x = y\ .$$

• T_1 *topologies.* We say that a space (X,τ) satisfies the *axiom* T_1 (or that it is a T_1*-space*) if

$$\forall\, x,\, y \in X,\, x \neq y\ \exists\, U \in \tau:\ x \in U \not\ni y\ . \qquad (T_1)$$

Exercise 9 X *is a* T_1*-space if and only if all finite sets in* X *are closed. That is, if and only if each point* $\{x\}$, $x \in X$, *is a closed set.*

• T_2 *topologies.* We say that a space (X,τ) satisfies the *axiom* T_2 (or that it is a T_2*-space* or that it is a *Hausdorff space*) if

$$\forall\, x,\, y \in X,\, x \neq y\ \exists\, U,\, V \in \tau:\ x \in U \wedge y \in V \wedge U \cap V = \emptyset\ . \qquad (T_2)$$

Proposition 10 (on continuous maps to T_2-spaces) *Let*

$$f, g\colon X \to Y$$

be continuous mappings and let Y be Hausdorff. Then the set

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in the space X.

Proof. We show that the complementary set

$$M := \{x \in X \mid f(x) \neq g(x)\}$$

is open in X. For any $x \in M$ we take two disjoint open sets $U_x \ni f(x)$ and $V_x \ni g(x)$ in Y. Then

$$M = \bigcup_{x \in M} f^{-1}[U_x] \cap g^{-1}[V_x]$$

and therefore M is an open set in X. □

A subset $M \subseteq X$ in a topological space is *dense (in X)* if $\overline{M} = X$. From the previous proposition we immediately obtain the next corollary.

Corollary 11 *Let*

$$f, g\colon X \to Y$$

be continuous mappings to a Hausdorff space Y and such that for a dense set $M \subseteq X$ one has that $f \,|\, M = g \,|\, M$. Then $f = g$.

The quasidiscrete topology of $(X, \leq)$ is T_0 iff $\leq$ is a partial order (and not just a preorder). With the exception of the discrete case, and this also holds for the Scott topology, it is not T_1. The cofinite topology is T_1 but if the underlying set is infinite, it is not Hausdorff. Metrizable topologies (and also the Sorgenfrey line) are Hausdorff; these are, however, much richer topologies that will be discussed in the following paragraphs.

Exercise 12 *Prove that the cofinite topology is T_1 but that, if the underlying set is infinite, it is not a Hausdorff topology.*

• *Zariski topology.* This is an important kind of topology used in algebraic geometry.

Theorem 13 (introducing Zariski topology) *Let $n \in \mathbb{N}$, K be a field, and $K[x_1, \dots, x_n]$ be the ring of polynomials in n variables x_i and with coefficients in K. For $P \subseteq K[x_1, \dots, x_n]$ we define*

$$Z(P) := \{\overline{a} \in K^n \mid \forall\, f \in P :\ f(\overline{a}) = 0_K\} \ .$$

Then

$$(K^n,\ \{Z(P) \mid P \subseteq K[x_1, \dots, x_n]\})$$

is a topology, defined by means of closed sets.

Proof. Clearly, $\emptyset$ and K^n are in the set system as $\emptyset = Z(\{1_K\})$ and $K^n = Z(\{0_K\})$; here 1_K and 0_K denote the corresponding constant polynomials. If P_i, $i \in J$, are subsets of $K[x_1, \dots, x_n]$ then it is easy to see that

$$\bigcap_{i \in J} Z(P_i) = Z(\bigcup_{i \in J} P_i) \ .$$

Thus the given set system is closed under arbitrary intersections. Finally, if $P, Q \subseteq K[x_1, \dots, x_n]$ then it follows that

$$Z(P) \cup Z(Q) = Z(\{fg \mid f \in P,\ g \in Q\}) \ .$$

Thus the given set system is closed under finite unions. □

Exercise 14 *Prove the last two equalities.*

Exercise 15 *Prove that Zariski topology is T_1.*

One can show that if the field K is infinite then every two nonempty open sets in Zariski topology intersect, and therefore it is not T_2.

• *T_3 topologies.* We say that a space (X, τ) satisfies the *axiom* T_3 (or that it is a *T_3-space* or that it is a *regular space*) if

$$\begin{aligned} &\forall\, x \in X\ \forall \text{ closed set } A \subseteq X \text{ with } x \notin A \\ &\exists\, U, V \in \tau :\ x \in U \wedge A \subseteq V \wedge U \cap V = \emptyset \ . \end{aligned} \qquad (T_3)$$

Theorem 16 (on regular spaces) *The following statements about a topological space $X = (X, \tau)$ are equivalent.*

1. *X is regular.*

2. *For every $x \in X$ and every neighborhood M of x there is a closed neighborhood N of x such that $N \subseteq M$.*

3. *For every $U \in \tau$,*

$$U = \bigcup \{V \mid V \in \tau \wedge \overline{V} \subseteq U\} .$$

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and $M \subseteq X$ be a neighborhood of x: there is a $W \in \tau$ such that $x \in W \subseteq M$. We have $x \notin X \setminus W$ and hence, by (1), there exist $U, V \in \tau$ such that $x \in U$, $X \setminus W \subseteq V$ and $U \cap V = \emptyset$. Set $N = \overline{U}$. Then indeed

$$x \in U \subseteq \overline{U} = N \subseteq X \setminus V \subseteq W \subseteq M .$$

(2) $\Rightarrow$ (3). Let $U \in \tau$. For every $x \in U$ choose by (2) to the neighborhood $x \in U$ an open set V_x and a closed set N_x such that

$$x \in V_x \subseteq N_x \subseteq U .$$

Then $\overline{V_x} \subseteq U$ and we have that $U = \bigcup_{x \in X} V_x$.

(3) $\Rightarrow$ (1). Suppose that $x \in X \setminus A$, where $A \subseteq X$ is a closed set. Using (3), take a $U \in \tau$ such that $x \in U$ and $\overline{U} \subseteq X \setminus A$. Then $U \ni x$ and $X \setminus \overline{U} \supseteq A$ are disjoint open sets. □

• $T_{3.5}$ *topologies.* We say that a space (X, τ) satisfies the *axiom* $T_{3.5}$ (or that it is a $T_{3.5}$-*space* or that it is a *completely regular space*) if ($[0, 1]$ is the real unit interval with the Euclidean topology)

$$\forall\, x \in X \ \forall \text{ closed set } A \subseteq X \text{ with } x \notin A$$
$$\exists \text{ a contin. map } f\colon X \to [0,1] :\ f(x) = 0 \wedge f[A] \subseteq \{1\} . \qquad (T_{3.5})$$

We easily see that a subset D of the interval $[0, 1]$ is dense iff for any two $a < b$ in $[0, 1]$ there exists a $d \in D$ such that $a < d < b$ (that is, iff it is dense in the sense used when speaking of order — and the same holds in each interval topology). Let (X, τ) be a topology. For open sets $U, V \in \tau$ we write

$$U <^* V \iff \exists \text{ dense } D \subseteq [0,\, 1] \text{ with } 0, 1 \in D \ \forall\, d \in D$$
$$\exists\, U_d \in \tau :\ U_0 = U \wedge U_1 = V \wedge (d, e \in D,\ d < e \Rightarrow \overline{U_d} \subseteq U_e) .$$

Theorem 17 (on completely regular spaces) *The following statements about a topological space $X = (X, \tau)$ are equivalent.*

1. *X is completely regular.*

2. *For every $x \in X$ and every open set $U \ni x$ there exists an open set $V \ni x$ such that $V <^* U$.*

3. *For every $U \in \tau$,*

$$U = \bigcup \{V \in \tau \mid V <^* U\} .$$

Proof. (1) ⇒ (2). Let $x \in U \in \tau$. Using (1), we take a continuous map $f \colon X \to [0,1]$ such that $f(x) = 0$ and $f[X \setminus U] \subseteq \{1\}$. We set $D := [0,1]$ and

$$U_a := f^{-1}[\,[0,\ (1+a)/2)\,],\ a \in [0,\ 1] .$$

Then $U_1 = U$. Clearly, $x \in U_0 <^* U_1 = U$.

The implication (2) ⇒ (1) follows from Proposition 17 about a continuous map in the last lecture.

That the equivalence (2) $\iff$ (3) holds is obvious.

□

• T_4 *topologies.* We say that a space (X, τ) satisfies the *axiom* T_4 (or that it is a T_4*-space* or that it is a *normal space*) if

$$\begin{aligned} &\forall \text{ closed sets } A, B \subseteq X \text{ with } A \cap B = \emptyset \\ &\exists\, U, V \in \tau :\ A \subseteq U \wedge B \subseteq V \wedge U \cap V = \emptyset . \end{aligned} \tag{T_4}$$

Theorem 18 (Urysohn's lemma) *A space X is normal if and only if for any two disjoint closed $A, B \subseteq X$ there is a continuous mapping $f \colon X \to [0,1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.*

Proof. Implication ⇐. Let $A, B \subseteq X$ be closed and disjoint. We take the described map f. Then, for example, $f^{-1}[\,[0,1/3)\,] \supseteq A$ and $f^{-1}[\,(2/3,1]\,] \supseteq B$ are disjoint open sets.

Implication ⇒. Separate A and B by open and disjoint sets U and V and set $U(0) := U$ and $U(1) := X \setminus B$. Suppose that the open sets $U(d)$ have been already defined for all $d = k/2^m$ with $m = 0, 1, \ldots, n$, $0 \le k \le 2^m$, such that $\overline{U(d)} \subseteq U(e)$ whenever $d < e$. Take the disjoint closed sets $\overline{U(k/2^n)}$ and $X \setminus U((k+1)/2^n)$, separate them by disjoint open sets $U \supseteq U(k/2^n)$ and $V \supseteq X \setminus U((k+1)/2^n)$, and set $U((k+1)/2^{n+1}) := U$. This way we obtain inductively, for all dyadic rational numbers $d \in [0,1]$, open sets $U(d)$ satisfying the assumption of Proposition 17 on a continuous map of the last lecture, and using it we get the required function separating A and B. $\square$

Exercise 19 *Prove that every metric space (M, ρ) yields a normal topology (M, τ) (we defined it earlier). Hint: separate disjoint closed sets $A, B \subseteq M$ by the function $f\colon M \to [0,1]$ given as*

$$f(x) = \frac{\rho(x,\, A)}{\rho(x,\, A) + \rho(x,\, B)}\ ,$$

where $\rho(x, A)$ is the distance of the point x from the set A.

Exercise 20 *Show that this metrizable topology (M, τ) is T_1.*

- *Separation axioms and standard constructions.* The sequence

$$T_0 \;\Leftarrow\; T_1 \;\Leftarrow\; T_2 \;\Leftarrow\; T_3 \wedge T_1 \;\Leftarrow\; T_{3.5} \wedge T_1 \;\Leftarrow\; T_4 \wedge T_1$$

is heading from general spaces to spaces of an increasingly "geometrical" nature (we will see shortly[1] that the spaces satisfying $T_{3.5} \wedge T_1$ look almost like subspaces of Euclidean spaces — with the difference that they can have an "infinite dimension").

None of the implications above can be inverted. For the first two we have already presented examples. To prove that a Hausdorff space is not necessarily regular is also very easy, and to see that complete regularity does not imply normality is not very hard either. But the relation of complete regularity and regularity had been a problem for quite a long time before it was solved.

Further, note the added requirements of T_1; without an extra assumption, the "higher" separation axioms would not imply the "lower" ones. In fact, to obtain T_1 from (complete) regularity it would suffice to add just T_0; normality plus T_0 does not imply T_1, though.

None of the separation properties is preserved under factorization (a trivial example: map $\mathbb{R}$ onto $\{0, 1\}$ by sending the rational numbers to 0 and the irrational ones to 1; then the quotient topology is not even T_0). On the other hand, the sums preserve all the T_i for trivial reasons.

The subspaces and products are more interesting.

[1] See Chapter V.5.9.

Theorem 21 (subspaces and products) *The axioms T_i, $i = 0$, 1, 2, 3 and 3.5 are preserved in subspaces and products.*

Proof. The cases of $i = 0, 1, 2$ are trivial. If $(x_i)_{i\in J} \neq (y_i)_{i\in J}$ in a product, we have for some k that $x_k \neq y_k$ in the space X_k; separate the points x_k and y_k by U, or by U and V, and use the sets $p_k^{-1}[U]$ and $p_k^{-1}[V]$.

Regularity and complete regularity. If X is (completely) regular, $Y \subseteq X$, $A \subseteq Y$ is closed in Y, and if $y \in Y$ is such that $y \notin A$, choose B closed in X such that $A = B \cap Y$. Then $y \notin B$ and if we separate y from B in X (by open sets U, V or by a real function f), then $U \cap Y$ and $V \cap Y$, resp. $f \,|\, Y$, separate y from the set A as required.

Now let X_i, $i \in J$, be regular (resp. completely regular), let $A \subseteq X = \prod_{i\in J} X_i$ be closed and let $x \in U := X \setminus A$. As U is open in X, there exist $i_1, \ldots, i_n \in J$ such that $x \in \bigcap_{j=1}^n p_{i_j}^{-1}[U_j]$ for some U_j open in X_{i_j}. In the regular case choose (by Theorem 16) sets V_j open in X_{i_j} and such that $x_{i_j} \in V_j \subseteq \overline{V_j} \subseteq U_j$. Set $V = \bigcap_{j=1}^n p_{i_j}^{-1}[V_j]$ and $W = X \setminus \bigcap_{j=1}^n p_{i_j}^{-1}[\overline{V_j}]$. Then $x \in V$; if $y \in A$ we have $y \notin \bigcap_{j=1}^n p_{i_j}^{-1}[U_j]$ and hence for some k we have $y_{i_k} \notin U_k$, and hence $y_{i_k} \notin \overline{V_k}$ and $y \in W$ and $A \subseteq W$. Obviously, $V \cap W = \emptyset$.

In the completely regular case choose continuous maps $f_j \colon X_{i_j} \to [0, 1]$, $j = 1, \ldots, n$, such that $f_j(x_{i_j}) = 0$ and $f_j[X_{i_j} \setminus U_j] \subseteq \{1\}$. Define $f \colon X \to [0, 1]$ by setting $f(y) = \max(f_1(y_{i_1}), \ldots, f_n(y_{i_n}))$. Obviously, $f(x) = 0$. If $y \in A$ then there exists a j such that $y_{i_j} \notin U_j$ and hence $f(y) = 1$. It is an easy exercise to prove that f is continuous. □

However, the normality is generally preserved neither in the subspaces nor in the product. We are not yet prepared for presenting counterexamples; they are not hard, though.

THANK YOU!

HOMEWORK: Exercises 14, 15, 19 and 20. Deadline is the end of the coming Sunday. Please, send me your solutions by e-mail to klazar@kam.mff.cuni.cz. To get credits for the tutorial, you should solve (or at least send in attempted solutions of) at least half of the homework exercises.