MATHEMATICAL STRUCTURES (NMAI064)
summer term 2021/22
lecturer: Martin Klazar

LECTURE 12 (May 9, 2022). TOPOLOGY: COMPACT SPACES
(based on the lecture notes of A. Pultr, Chapter V.6)

• *Compactness.* A *cover* of a topological space (X,τ) is any subset $\mathcal{U}\subseteq\tau$ such that $\bigcup\mathcal{U}=X$. For $Y\subseteq X$, a *cover of* Y is any subset $\mathcal{U}\subseteq\tau$ such that $\bigcup\mathcal{U}\supseteq Y$. The space (X,τ) is *compact* if

$$\forall\text{ cover }\mathcal{U}\ \exists\text{ finite }\mathcal{V}\subseteq\mathcal{U}:\ \bigcup\mathcal{V}=X\ .$$

We say that every cover (of the space X) has a finite subcover. Similarly, $Y\subseteq X$ is a compact subset if every cover of Y has a finite subcover of Y.

Exercise 1 *Let $Y\subseteq X$ where (X,τ) is a topological space. Prove that Y is compact if and only if the subspace $(Y,\tau\,|\,Y)$ is compact.*

Exercise 2 *Show that every finite discrete topological space is compact.*

Theorem 3 (two properties of compactness) *These are as follows.*

1. *Every closed subset in a compact space is compact.*
2. *Continuous image of a compact set is always compact.*

Proof. 1. So let (X,τ) be a compact space, $Y\subseteq X$ be a closed set and $\mathcal{U}\subseteq\tau$ be a cover of Y. Then $\mathcal{U}\cup\{X\setminus Y\}$ is a cover of X. We take from it a finite subcover $\mathcal{V}$ of X. Clearly, $\mathcal{V}\setminus\{X\setminus Y\}\subseteq\mathcal{U}$ is a finite subcover of Y.

2. So let $f\colon(X,\tau)\to(Y,\sigma)$ be a continuous map, defined on a compact space X (cf. Exercise 1), and let $\mathcal{U}\subseteq\sigma$ be a cover of $f[X]$. Then

$$\{f^{-1}[U]\mid U\in\mathcal{U}\}$$

is a cover of X. Since X is compact, there exists a finite subset $\mathcal{V} \subseteq \mathcal{U}$ such that

$$\{f^{-1}[U] \mid U \in \mathcal{V}\}$$

is still a cover of X. Since $f[f^{-1}[U]] = U$, it follows that $\mathcal{V} \subseteq \mathcal{U}$ is a finite subcover of $f[X]$. $\square$

Exercise 4 *Can an open set be compact?*

• *Alexander's lemma.* This is a useful result: to prove that a topology is compact, it suffices to check the condition on covers only for any subbasis.

Theorem 5 (Alexander's lemma) *Suppose that (X, τ) is a topological space and that $\mathcal{S} \subseteq \tau$ is a subbasis such that every cover $\mathcal{U} \subseteq \mathcal{S}$ has a finite subcover. Then X is compact.*

Proof. We proceed by contradiction; thus we assume that X is not compact but that every cover taken from a subbasis $\mathcal{S}$ has a finite subcover. We say that a cover $\mathcal{U} \subseteq \tau$ is *large* if it has no finite subcover. By our assumption there exists a large cover. We prove:

> there exist an inclusion-wise maximal large cover $\mathcal{A}$; that is, a large cover $\mathcal{A}$ such that by adding any $U \in \tau \setminus \mathcal{A}$ to $\mathcal{A}$ yields a cover that has a finite subcover.

This follows by a standard argument using Zorn's lemma; we review it below in Exercise 6. So we take the set of all large covers and the order by inclusion on it, and check that the upper bound assumption of Zorn's lemma is satisfied. Suppose that Y is a chain of large covers. We claim that $\mathcal{U}' := \bigcup Y$ is a large cover, which is then an upper bound of Y. Indeed, if $\mathcal{U}'$ had a finite subcover $\mathcal{V} \subseteq \mathcal{U}'$, there would be a $\mathcal{U}'' \in Y$ such that $\mathcal{V} \subseteq \mathcal{U}''$ (because Y is a chain and in any chain any finite subset has a largest element), in contradiction with the fact that $\mathcal{U}''$ is a large cover. Thus the hypothesis of Zorn's lemma is satisfied in our situation and, by Zorn's lemma, there exists a maximal large cover $\mathcal{A}$.

We set $\mathcal{B} := \tau \setminus \mathcal{A}$. Then for all $U \in \tau$,

$$U \in \mathcal{B} \iff \exists\, U_1, \ldots, U_n \in \mathcal{A}: \; U \cup U_1 \cup \cdots \cup U_n = X \; .$$

The implication $\Rightarrow$ follows from the maximality of $\mathcal{A}$, and the opposite one $\Leftarrow$ from the fact that $\mathcal{A}$ is a large cover. We show that the set system $\mathcal{B}$ is closed under taking supersets and finite intersections. Indeed, if $V, U \in \tau$ are such that $V \supseteq U \in \mathcal{B}$, then $U \cup U_1 \cup \cdots \cup U_n = X$ for some $U_i \in \mathcal{A}$, thus also $V \cup U_1 \cup \cdots \cup U_n = X$ and $V \in \mathcal{B}$; and if $U_1, U_2 \in \mathcal{B}$ then $U_1 \cup U_1' \cup \cdots \cup U_n' = X$ and $U_2 \cup U_1'' \cup \cdots \cup U_m'' = X$ for some $U_i', U_j'' \in \mathcal{A}$, but then

$$X \setminus \left(\bigcup_{i=1}^n U_i' \cup \bigcup_{j=1}^m U_j''\right) \subseteq U_1 \cap U_2$$

and $U_1 \cap U_2 \in \mathcal{B}$.

Now we finish the proof by deriving a contradiction from our assumptions. For any $x \in X$ there is a $U \in \mathcal{A}$ such that $x \in U$, because $\mathcal{A}$ is a cover. Thus there are $S_1, \ldots, S_n \in \mathcal{S}$ such that

$$x \in \bigcap_{i=1}^{n} S_i \subseteq U \in \mathcal{A}$$

because $\mathcal{S}$ is a subbasis. By the above closure properties of $\mathcal{B} = \tau \setminus \mathcal{A}$ we see that not all S_i are in $\mathcal{B}$. Thus some $S_i = S_{i(x)} \in \mathcal{A}$. By our assumption on $\mathcal{S}$ the cover

$$\{S_{i(x)} \mid x \in X\}$$

has a finite subcover. But it is also a subcover of $\mathcal{A}$, in contradiction with the fact that $\mathcal{A}$ is large. □

Exercise 6 (Zorn's lemma) *Recall that this is a result on orders $(X, \leq_X)$ that is equivalent to the axiom of choice and says the following. If every chain $Y \subseteq X$ (i.e., $\leq_X$ restricted to Y is a liner order) has an upper bound (an $x \in X$ such that $y \leq_X x$ for every $y \in Y$), then for every $x \in X$ there is a maximal element $x' \in X$ with $x \leq_X x'$ (the maximality means that $x' <_X x''$ for no $x'' \in X$).*

Corollary 7 (on intervals) *Every interval $[a, b]$ is compact (in the Euclidean topology).*

Proof. We assume that $a < b$ (for $a \geq b$ the result is trivial) and check the cover condition for the subbasis

$$\mathcal{S} = \{[a,\, c) \mid c \in (a,\, b)\} \cup \{(c,\, b] \mid c \in (a,\, b)\}$$

of the Euclidean topology on $[a, b]$. Let $\mathcal{U} \subseteq \mathcal{S}$ be a cover and let

$$d := \sup(\{c \in (a,\, b) \mid [a,\, c) \in \mathcal{U}\}) \in [a,\, b]$$

(the set is nonempty as a cannot be covered by any interval $(c, b]$). The point d has to be covered by some interval $(c, b] \in \mathcal{U}$, hence $c < d$. By the definition of supremum there is a $c' \in (a, b)$ such that $[a, c') \in \mathcal{U}$ and $c < c'$. We got a finite (actually two-element) subcover

$$[a,\, c') \cup (c,\, b] = [a,\, b]$$

of $\mathcal{U}$. □

• *Tikhonov's (Tichonov's, Tychonoff's, ...) theorem.* This is one the most important and useful results on compact spaces, or even in the whole topology.

Theorem 8 (A. N. Tikhonov, 1935) *Every product of compact topological spaces is a compact space.*

Proof. The proof is in fact easy; we again apply Alexander's lemma. Let (X_i, τ_i), $i \in J$, be compact spaces and let $\mathcal{U}$ be a cover of $\prod_{i \in J} X_i$, taken from the definitoric subbasis

$$\mathcal{S} = \{p_i^{-1}[U] \mid i \in J,\, U \in \tau_i\}\ .$$

For any $i \in J$ we set

$$\mathcal{U}_i := \{U \in \tau_i \mid p_i^{-1}[U] \in \mathcal{U}\}\ .$$

We claim that there is a $k \in J$ such that $\mathcal{U}_k$ covers X_k. For if not, then we could choose for every $i \in J$ a point $x_i \in X_i \setminus \bigcup \mathcal{U}_i$, but then the point $(x_i)_{i \in J}$ in the product would not be covered by $\mathcal{U}$. Since X_k is compact, we have a finite subcover $\mathcal{V} \subseteq \mathcal{U}_k$. Then

$$\{p_i^{-1}[U] \mid U \in \mathcal{V}\}$$

is a finite subcover of $\mathcal{U}$ (of the product space). □

Wikipedia says on the theorem that it was first stated and proved by *Andrey Nikolayevich Tikhonov (1906–1993)* (who besides being a mathematician was also a geophysicist) in a particular case in 1935, and that for the general form of the theorem the "earliest known published proof is contained in a 1937 paper of Eduard Čech". *Eduard Čech (1893–1960)* was a world-famous Czech mathematician, mainly topologist.

We present an application of Tikhonov's theorem on proper colorings of infinite graphs. We begin by restating the definition of compactness; we leave it to you as an exercise.

Exercise 9 *A topological space* (X, τ) *is compact if and only if every system* $\{A_i \mid i \in J\}$ *of closed sets in* X *has the* finite intersections property*:*

$$\big(\forall \textit{ finite } I \subseteq J : \ \textstyle\bigcap_{i\in I} A_i \neq \emptyset\big) \Rightarrow \textstyle\bigcap_{i\in J} A_i \neq \emptyset \ .$$

For a graph $G = (V, E)$, so $E \subseteq \binom{V}{2}$, its *proper* X*-coloring* is any map $f\colon V \to X$ such that $f(u) \neq f(v)$ whenever $\{u, v\} \in E$. Without the last restriction, the map f is called simply an X*-coloring (of* G*)*. The *chromatic number* $\chi(G) \in \mathbb{N}$ *of* G is defined as

$$\chi(G) := \min(\{k \in \mathbb{N} \mid \exists \text{ proper } [k]\text{-coloring of } G\}) \ ,$$

where $[k] = \{1, 2, \ldots, k\}$. For example, the 5-cycle

$$C_5 := (\{1,\ 2,\ 3,\ 4,\ 5\},\ \{\{1,\ 2\},\ \{2,\ 3\},\ \{3,\ 4\},\ \{4,\ 5\},\ \{5,\ 1\}\})$$

has $\chi(C_5) = 3$. We say that a graph $G' = (V', E')$ is a *subgraph* of another graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

Theorem 10 (compactness of χ) *Let* $k \in \mathbb{N}$ *and let* $G = (V, E)$ *be a graph, where the vertex set* V *may be arbitrary (i.e., of any cardinality). Then*

$$\chi(G) \leq k \iff \forall \textit{ finite subgraph } H \textit{ of } G : \ \chi(H) \leq k \ .$$

Proof. In other words, G has a proper $[k]$-coloring if and only if every finite subgraph of G has it too. The "only if" part is clear, and we have to prove: if every finite subgraph H of G has a proper $[k]$-coloring, then G has it too. For every $v \in V$ we take a copy $X_v := ([k], \mathcal{P}([k]))$ of the discrete space on $[k]$ (where every subset of $[k]$ is open and closed) and consider the product space

$$X = (X,\ \tau) := \prod_{v\in V} X_v \ .$$

Note that every point in it, $\overline{x} := (x_v)_{v\in V}$ with $x_v \in [k]$, is a $[k]$-coloring of G. For any edge $e = \{u, v\} \in E$ of G we consider the set

$$A_e := \{\overline{x} \in X \mid x_u \neq x_v\} \ .$$

These are exactly the $[k]$-colorings of G that restrict to proper $[k]$-colorings of the one-edge subgraph $(\{u, v\}, \{\{u, v\}\})$ of G. Since every space X_v is discrete and by the definition of the product topology τ, it is clear that every set A_e is closed in X (Exercise 13). Since every finite subgraph of G has a proper $[k]$-coloring, it follows that the hypothesis of the implication in Exercise 9 is satisfied in X for the system $\{A_e \mid e \in E\}$ (note that a proper coloring of a subgraph trivially extends to a coloring of the whole graph). Since X is compact by Exercise 2 and Tikhonov's theorem, by Exercise 9 we have that

$$\bigcap_{e\in E} A_e \neq \emptyset$$

and the whole graph G has a proper $[k]$-coloring. □

Exercise 11 *Prove the previous equivalence directly, without using Tikhonov's theorem, for* $k = 1$.

Exercise 12 *Prove the previous equivalence directly, without using Tikhonov's theorem, for* $k = 2$.

Exercise 13 *Explain why the sets* A_e *in the previous proof are closed.*

- *Compactness and Hausdorffness.*

Theorem 14 (compacts in Hausdorff spaces) *Compact sets in Hausdorff spaces (i.e.,* T_2*-spaces) are closed.*

Proof. Let $Y \subseteq X$ be a compact set in a Hausdorff space (X, τ). It suffices to show that for every point $x \in X \setminus Y$ there exists a $U_x \in \tau$ such that $x \in U_x \subseteq X \setminus Y$. For then $X \setminus Y = \bigcup_{x\in X\setminus Y} U_x$ is open and Y is closed. We fix an $x \in X \setminus Y$ and take for any $y \in Y$ sets $U_{x,y}, V_y \in \tau$ such that

$$x \in U_{x,y} \wedge y \in V_y \wedge U_{x,y} \cap V_y = \emptyset \ .$$

This is possible because X is a Hausdorff space. Since $\{V_y \mid y \in Y\}$ is a cover of Y and Y is compact, there exist finitely many points $y_1, \ldots, y_n \in Y$ such that $\{V_{y_i} \mid i = 1, 2, \ldots, n\}$ is a cover of Y. We set

$$U_x := \bigcap_{i=1}^{n} U_{x, y_i} .$$

This set is as required: it is open, contains x and is disjoint to Y because it is disjoint even to the superset $\bigcup_{i=1}^{n} V_{y_i}$ of Y. □

Corollary 15 (on inverse maps) *The inverse map to a continuous injection from a compact space to a Hausdorff space is continuous.*

Proof. Suppose that (X, τ) is a compact space, (Y, σ) is a Hausdorff space and $f\colon X \to Y$ is a continuous injection. Then for every closed set $A \subseteq X$ we have that

$$(f^{-1})^{-1}[A] = f[A] \subseteq Y .$$

By part 1 of Theorem 3, the set A is compact and by part 2 of the same theorem, $f[A]$ is compact too. By the previous theorem, $f[A]$ is closed. Thus the inverse image of any closed set in X by the map f^{-1} is closed. Hence f^{-1} is continuous. □

Theorem 16 (compact Hausdorff is ...) *Every compact Hausdorff space (X, τ) is normal (i.e., a T_4-space).*

Proof. Let $A \subseteq X$ be a closed set and $x \in X \setminus A$ be a point. By part 1 of Theorem 3, the set A is compact. Arguing like in the proof of the previous theorem, we obtain open sets $U, V \in \tau$ such that $x \in U$, $A \subseteq V$ and $U \cap V = \emptyset$ (Exercise 17). Now let $A, B \subseteq X$ be two disjoint closed sets. Using the previous step, we take for any $x \in A$ sets $U_x, V_x \in \tau$ such that

$$x \in U_x \wedge B \subseteq V_x \wedge U_x \cap V_x = \emptyset .$$

By part 1 of Theorem 3, the set A is compact. We select a finite subcover from the cover $\{U_x \mid x \in A\}$ of A and get points $x_1, \ldots, x_n \in A$ such that A is covered by the U_{x_i}, $i = 1, 2, \ldots, n$. Then

$$\bigcup_{i=1}^{n} U_{x_i} \supseteq A \quad \text{and} \quad \bigcap_{i=1}^{n} V_{x_i} \supseteq B$$

are disjoint open sets separating A and B. $\square$

Exercise 17 *Explain in detail how we get in the previous proof the open sets U and V.*

• *Locally compact spaces and Lindelöf spaces.* A topological space (X, τ) is *locally compact*, if for every point $x \in X$ and every open set $U \ni x$ there is a compact neighborhood K of x such that $x \in K \subseteq U$. That is, there is an open set V and a compact set K such that

$$x \in V \subseteq K \subseteq U \ .$$

Local compactness is sufficient in many situations when the whole space is not compact and plays an important role in mathematics.

Exercise 18 *Is the Euclidean interval* $(0, 1)$ *locally compact?*

Compact space need not be locally compact, but we have the following result.

Exercise 19 *Show that every compact Hausdorff space is locally compact.*

A topological space (X, τ) is a *Lindelöf space* if every cover has an at most countable subcover.

Theorem 20 (regular Lindelöf is ...) *Every regular space (i.e., a T_3-space) that is Lindelöf is normal (i.e., a T_4-space).*

Proof. We suppose that (X, τ) is a regular Lindelöf space and that $A, B \subseteq X$ are two disjoint closed sets; we will separate them by disjoint open sets U and V.

Using regularity of X, we separate any point $x \in A$ and the set B by two disjoint open sets. Thus we get a set $U_x \in \tau$ such that

$$x \in U_x \subseteq \overline{U_x} \subseteq X \setminus B \ .$$

Using Exercise 21, we select from the cover $\{U_x \mid x \in A\}$ of A an at most countable subcover $U_1, U_2, \dots$ of A. We may assume that $U_1 \subseteq U_2 \subseteq \dots$ (by replacing these sets by the unions $U_1, U_1 \cup U_2, \dots$). Thus

$$U_1 \subseteq U_2 \subseteq \cdots \wedge A \subseteq \bigcup_{i=1}^{\infty} U_i \wedge \forall\, i : \ \overline{U_i} \cap B = \emptyset \ .$$

Similarly we get sets $V_i \in \tau$ such that

$$V_1 \subseteq V_2 \subseteq \cdots \wedge B \subseteq \bigcup_{i=1}^{\infty} V_i \wedge \forall i: \ \overline{V_i} \cap A = \emptyset \ .$$

Let

$$U := \bigcup_{i=1}^{\infty} \underbrace{U_i \cap (X \setminus \overline{V_i})}_{U_i'} \ \text{ and } \ V := \bigcup_{i=1}^{\infty} \underbrace{V_i \cap (X \setminus \overline{U_i})}_{V_i'} \ .$$

It is easy to see that these are the required sets. Namely, $U, V \in \tau$, $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$, because for every pair of indices i, j one has that $U_i' \cap V_j' = \emptyset$ (since the sequences of sets (U_i) and (V_i) increase, for $i \leq j$ one has that $U_i \cap (X \setminus \overline{U_j}) = \emptyset$, and for $j \leq i$ similarly that $V_j \cap (X \setminus \overline{V_i}) = \emptyset$). □

Exercise 21 *Prove that any closed subspace of a Lindelöf space is a Lindelöf space.*

THANK YOU!

HOMEWORK: Exercises 1, 9, 12 and 21. *This is the last set of homework exercises.* Deadline is (by the end of the day) May 14, 2022. To get "zápočet" for the tutorial, you should solve (or at least send in solutions of) at least half of the homework exercises.