Solutions to HW 10

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1. The function $f\colon [0,1] \to \mathbb{R}$, $f(\frac{1}{n}) = 1$ for $n \in \mathbb{N}$ and $f(x) = 0$ for other $x \in [0,1]$, is Riemann-integrable by the Lebesgue theorem because it is bounded (its image is just $\{0,1\}$) and the set of points of discontinuity of f is the countable set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ which has measure 0 (can be covered by intervals with the total length $\frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots = \varepsilon$).

Every lower sum $s(f,P) = 0$ because $\inf_{x \in I} f(x) = 0$ for any nontrivial interval $I \subset [0,1]$. Hence $\underline{\int_0^1} f = 0$ (it is the supremum of lower sums). We show that for any $\delta > 0$ there is a partition P of $[0,1]$ for which the upper sum $S(f,P) = \delta$. This shows that also $\overline{\int_0^1} f = 0$ (it is the infimum of upper sums). Hence $\int_0^1 f = 0$.

For the given δ we define the partition $P = (0 = a_0 < a_1 < \cdots < a_n = 1)$ in the following way. We set $a_1 := \delta/2$ and take the other points a_i, $1 < i < n$, so that every subinterval $I_i := [a_{i-1}, a_i]$ contains at most one reciprocal $\frac{1}{n}$, $n \in \mathbb{N}$, with $a_1 < \frac{1}{n} \le a_n = 1$ and that the total length of the subintervals $\{I_i \mid i \in X\}$ containing these (finitely many!) reciprocals is $\delta/2$. It is easy to see that this can be done and that then

$$\begin{aligned} S(f,\,P) &= (a_1 - a_0)\sup_{x \in I_1} f(x) + \sum_{i=2}^{n} (a_i - a_{i-1}) \sup_{x \in I_i} f(x) \\ &= \frac{\delta}{2} \cdot 1 + \sum_{i=2,\, i \notin X}^{n} |I_i| \cdot 0 + \sum_{i \in X} |I_i| \cdot 1 = \frac{\delta}{2} + 0 + \frac{\delta}{2} = \delta\;. \end{aligned}$$

2. The simplest proof uses the Lebesgue theorem. If $f, g \in \mathcal{R}(a,b)$ then both functions are bounded, $|f|, |g| \le c$ on $[a,b]$, and their sets of discontinuities $\mathrm{DC}(f)$ and $\mathrm{DC}(g)$ have measure 0. Thus $f+g$ is bounded too, $|f+g| \le 2c$ on $[a,b]$, and $\mathrm{DC}(f+g)$ has measure 0 too because $\mathrm{DC}(f+g) \subset \mathrm{DC}(f) \cup \mathrm{DC}(g)$. Thus, by the L. theorem, $f + g \in \mathcal{R}(a,b)$.

It is not hard to prove this also by the Darboux definition of the R. integral (lower and upper sums), and in fact to prove in this way more strongly that $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

3. The proof via the L. theorem is the same as in 2. Again one can prove it by the Darboux definition, but now it is more complicated: one first proves (by

lower and upper sums) that $f \in \mathcal{R}(a,b) \Rightarrow f^2 \in \mathcal{R}(a,b)$ and then one uses the formula $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$.

4. See Chapter XVI in prof. A. Pultr's lecture notes

https://kam.mff.cuni.cz/~pultr/analysiscourse.pdf

5. If (M,d) and (N,e) are metric spaces then a map $f\colon M \to N$ is uniformly continuous if (and only if)

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0\, \big(x,\, y \in M,\, d(x,\, y) < \delta \Rightarrow e(f(x),\, f(y)) < \varepsilon\big)\ .$$