Supplementary material for tutorial 8 (2024) of MA2

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November 20, 2024

Here are the results needed for solving the first three HW problems.

Proposition 0.1 (Taylor polynomials) *Let $m, n \in \mathbb{N}$, $a \in U \subset \mathbb{R}^m$ where U is an open set and let*

$$f = f(x_1,\,\dots,\,x_m)\colon U \to \mathbb{R}$$

be a map that has on U continuous partial derivatives up to the order n. Then for $h = (h_1, \dots, h_m) \in \mathbb{R}^m$ with $\|h\| \to 0$ we have the expansion

$$f(a+h) = \sum_{\substack{i_1,\,\dots,\,i_m \ge 0\\ i_1+\dots+i_m \le n}} \frac{1}{i_1!\dots i_m!} \cdot \frac{\partial^{i_1+\dots+i_m} f}{\partial x_1^{i_1}\dots \partial x_m^{i_m}}(a) \cdot h_1^{i_1}\dots h_m^{i_m} + o(\|h\|^n)\ .$$

We call it the Taylor polynomial of f with center in a and order n.

The formal series

$$\sum_{i_1,\,\dots,\,i_m \ge 0} \frac{1}{i_1!\dots i_m!} \cdot \frac{\partial^{i_1+\dots+i_m} f}{\partial x_1^{i_1}\dots \partial x_m^{i_m}}(a) \cdot h_1^{i_1}\dots h_m^{i_m}$$

— to define it it suffices that all partial derivatives of f are defined in a — is called the *Taylor expansion of f (with center in a)*.

For a function $f = f(x_1, \dots, x_m)$ and $b \in \mathbb{R}^m$, the *gradient* $\nabla_f(b)$ is simply the (row) vector of the values of its order 1 partial derivatives in b,

$$\nabla_f(b) = \big(\partial f/\partial x_1(b),\, \partial f/\partial x_2(b),\, \dots,\, \partial f/\partial x_m(b)\big)\ .$$

We define the $m \times m$ real matrix

$$H_f(b) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(b)\right)_{i,\,j=1}^{m}\ .$$

We know that in some situations, certainly in the following theorem, this matrix is symmetric. If $M \in \mathbb{R}^{m\times m}$ is a symmetric matrix, we define a quadratic form

$$P_M(x_1,\,\dots,\,x_m) := (x_1,\,\dots,\,x_m) \cdot M \cdot (x_1,\,\dots,\,x_m)^T\ .$$

For example, if

$$N := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } N' := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then $P_N(x, y) = yx + xy = 2xy$ and $P_{N'}(x, y) = x^2 + y^2$. We say that M is

1. *positively definite* if $P_M(x_1, \ldots, x_m) > 0$ for every $(x_1, \ldots, x_m) \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\}$,
2. *negatively definite* if $P_M(x_1, \ldots, x_m) < 0$ for every $(x_1, \ldots, x_m) \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\}$ and
3. *indefinite* if $P_M(x_1, \ldots, x_m) > 0$ and $P_M(y_1, \ldots, y_m) < 0$ for some two points $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ in $\mathbb{R}^m$.

Definiteness of M can be determined by expressing the form $P_M(x_1, \ldots, x_m)$ as a linear combination of m squares. If all m coefficients in this combination are > 0 (resp. < 0), M is positively (resp. negatively) definite. If there are both positive and negative coefficients, M is indefinite. For example,

$$P_N(x, y) = 2xy = \tfrac{1}{2}(x + y)^2 - \tfrac{1}{2}(x - y)^2$$

and N is indefinite, but this is obvious already from $P_N(x, y) = 2xy$. Of course, $P_{N'}(x, y) = x^2 + y^2$ is positively definite. If there are zeros among the coefficients and M is not indefinite, then M is neither positively nor negatively definite.

Let $b \in M \subset \mathbb{R}^m$ and $f \colon M \to \mathbb{R}$. We say that f has in the point b *local minimum* (resp. *local maximum*) if there is an $r > 0$ such that $f(b) \le f(x)$ (resp. $f(b) \ge f(x)$) for every $x \in B(b, r) \cap M$. If the inequality holds as strict ($<$, resp. $>$) for every $x \in B(b, r) \cap M$, $x \ne b$, we say that the local extreme is *strict*.

Theorem 0.2 (criteria for extremes) *Let $m \in \mathbb{N}$, $b \in U \subset \mathbb{R}^m$ where U is an open set and let*

$$f = f(x_1, \ldots, x_m) \colon U \to \mathbb{R}$$

be a map that has on U continuous partial derivatives up to the order 2. Then the following hold.

1. *If the gradient $\nabla_f(b) \ne (0, 0, \ldots, 0)$ then f does not have in the point b local extreme.*
2. *If the gradient $\nabla_f(b) = (0, 0, \ldots, 0)$ and the matrix $H_f(b)$ is positively (resp. negatively) definite, then f has in the point b a strict local minimum (resp. maximum).*
3. *If the gradient $\nabla_f(b) = (0, 0, \ldots, 0)$ and the matrix $H_f(b)$ is indefinite, then f does not have in the point b local extreme.*