EXERCISES FOR TUTORIAL 6 OF MA 2, Nov 6, 2024

In the following $M = (M, d)$ is a metric space. We say that a subset $X \subset M$ is complete if the subspace (X, d) is complete, that is, every Cauchy sequence in X has a limit in X.

1. Is the intersection of two complete sets in M always a complete set?
2. Is the union of two complete sets in M always a complete set?
3. Let $f\colon A \to \mathbb{R}$ be a real function, defined on an arbitrary set A. Write formally by a formula, using quantifiers, that f has exactly one minimum value.
4. Let $F(x, y) = \sin(x + y) - \frac{1}{2}$. For which points $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0, y_0) = 0$ is the assumption of the theorem on implicit functions (TIF) satisfied, so that we can solve the equation $F(x, y) = 0$ for $y = f(x)$ in a neighborhood of x_0? Compute $f'(x_0)$ in two ways: using the formula in TIF and then directly (find $f(x)$ explicitly and differentiate it).
5. The same for the variable y, that is, for the function $x = g(y)$.