MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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LECTURE 9 (April 12, 2023) INTRODUCTION TO COMPLEX ANALYSIS 1

• *Complex numbers.* In this and two following lectures we will prove Theorem 6 given below: if a function $f\colon \mathbb{C} \to \mathbb{C}$ has everywhere derivative, it is the sum of a power series, i.e. there exist complex coefficients $a_0, a_1, \dots$ such that for every $z \in \mathbb{C}$ one has that $f(z) = \sum_{n\ge 0} a_n z^n$. *Complex numbers*

$$\mathbb{C} = \{z = a + bi \mid a,\, b \in \mathbb{R}\},\ \ i = \sqrt{-1}\ ,$$

form the normed field $(\mathbb{C}, 0, 1, +, \cdot, |\cdots|)$ (see earlier), with the norm $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

Exercise 1 *Prove for complex numbers the triangle inequality that* $\forall\, u, v \in \mathbb{C}\,\big(|u + v| \le |u| + |v|\big)$.

It is also a metric space $(\mathbb{C}, d)$ with the metric $d(z_1, z_2) = |z_1 - z_2|$. It is isometric to the classical Euclidean plane $\mathbb{R}^2$ and is complete.

Exercise 2 *Prove that* $(\mathbb{C}, d)$ *is a complete metric space.*

The symbols $U, U_0, U_1, \dots$ denote non-empty open subsets of $\mathbb{C}$, and z is a complex variable. Recall the notation

$$\mathrm{re}(a + bi) := a \ \text{ and }\ \mathrm{im}(a + bi) := b$$

for the real and imaginary part of the number $a + bi$. For a given $u \in \mathbb{C}$ and $r > 0$, we denote by

$$B(u,\, r) = \{z \in \mathbb{C} \mid |z - u| < r\}$$

the ball (open disc) with the center u and radius $r > 0$.

• *Holomorphic and analytic functions.* For a function $f\colon U \to \mathbb{C}$ and a point $z_0 \in U$, the *derivative* $f'(z_0)$ of f at the point z_0 is defined as for real functions:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}\,,$$

if this limit exists. Explicitly speaking, the number $f'(z_0) \in \mathbb{C}$ is the derivative of the function f at the point z_0 if and only if

$$\forall\,\varepsilon > 0\ \exists\,\delta > 0\,\big(z \in U \wedge 0 < |z - z_0| < \delta \Rightarrow \\ \Rightarrow \left|\frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)\right| < \varepsilon\bigg)\,.$$

A function $f\colon U \to \mathbb{C}$ is *holomorphic (on U)* if it has derivative at every point $z_0 \in U$. An *entire* function $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic on the entire complex plane $\mathbb{C}$. Complex derivatives have the same algebraic properties as real derivatives. We leave their proofs as an exercise.

Proposition 3 (properties of derivatives) *Let*

$$f,\, g\colon U \to \mathbb{C} \ \ \textit{and} \ \ h\colon U_0 \to \mathbb{C}$$

be holomorphic functions and $\alpha, \beta \in \mathbb{C}$. The following hold.

1. *The function $\alpha f + \beta g$ is holomorphic on U and $(\alpha f + \beta g)' = \alpha f' + \beta g'$.*
2. *The product fg is holomorphic on U and $(fg)' = f'g + fg'$.*
3. *If $g \neq 0$ on U, then the fraction f/g is holomorphic on U and $(f/g)' = (f'g - fg')/g^2$.*

4. If $h[U_0] \subset U$, then the composite function $f(h)\colon U_0 \to \mathbb{C}$ is holomorphic on U_0 and $(f(h))' = f'(h) \cdot h'$.

Exercise 4 *Prove this proposition.*

Furthermore, as for real functions, for $n \in \mathbb{N}$ we have that $(z^n)' = nz^{n-1}$ on $\mathbb{C}$, the derivative of a constant function is the zero function and every rational function is holomorphic on its definition domain and its derivative is the same as in the real case (it is given by the same formula).

The function $f\colon U \to \mathbb{C}$ is *analytic (on U)* if for each point $z_0 \in U$ there exist complex numbers $a_0, a_1, \dots$ such that

$$z \in U \wedge B(z_0,\, |z - z_0|) \subset U \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

– in every disc with the center z_0 contained in U, the analytic function is expressed by a power series with complex coefficients and center z_0. We compute with power series with complex coefficients in exactly the same way as with real power series.

Exercise 5 *Prove that every analytic function $f\colon U \to \mathbb{C}$ is holomorphic on U.*

• *Four differences of complex and real analysis.*

The first difference is the next theorem.

Theorem 6 (holomorphic $\Rightarrow$ analytic) *If*

$$f\colon \mathbb{C} \to \mathbb{C}$$

is an entire function, then there exist complex coefficients $a_0, a_1, \dots$

such that for each $z \in \mathbb{C}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \; .$$

For the sake of simplicity, in these lectures we prove only this version for entire functions. However, it is true that if f is holomorphic on U then f is analytic on U. For real functions this does not hold.

Exercise 7 *Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined for $x \le 0$ as 0 and for $x \ge 0$ as $f(x) = x^2$. Prove that* (i) *for every $x \in \mathbb{R}$ there is a finite derivative $f'(x)$, but that* (ii) *there are no real (or complex) numbers a_n such that $f(x) = \sum_{n\ge 0} a_n x^n$ (in a neighborhood of 0). Hint for* (ii): *show that sum of a power series has derivatives of all orders.*

The second difference. $f\colon U \to \mathbb{C}$ is *bounded* when $\exists\, c > 0\; \forall\, z \in U\; \big(|f(z)| < c\big)$. We also prove the following theorem.

Theorem 8 (J. Liouville, 1847) *If $f\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.*

This is also not true for real functions.

Exercise 9 *Prove that the function $f(x) := \mathrm{e}^{-x^2}\colon \mathbb{R} \to \mathbb{R}$ is a counterexample to the real Liouville theorem.*

Exercise 10 *Derive from Liouville's Theorem the Fundamental Theorem of Algebra which states that every non-constant complex polynomial $p(z)$ has a root. Hint: consider the (entire?) function $1/p(z)$.*

The third difference between real and complex analysis concerns the continuity of the derivative.

Corollary 11 ($\forall$ derivatives) *Every holomorphic function*

$$f\colon U \to \mathbb{C}$$

has derivatives $f^{(n)}(z)$ of all orders $n \in \mathbb{N}$. In particular, its derivative $f'\colon U \to \mathbb{C}$ is a continuous function.

Proof. This follows from the fact that any holomorphic function is analytic. $\square$

Exercise 12 *Describe a function $f\colon \mathbb{R} \to \mathbb{R}$ that has derivative $f'\colon \mathbb{R} \to \mathbb{R}$ but does not have the second order derivative $f''\colon \mathbb{R} \to \mathbb{R}$.*

Exercise 13 *Describe a function $f\colon \mathbb{R} \to \mathbb{R}$ with discontinuous derivative $f'\colon \mathbb{R} \to \mathbb{R}$. Hint: we saw such function in Mathematical Analysis 1.*

The fourth difference of real and complex analysis is perhaps the most surprising.

Theorem 14 (maximum modulus principle) *Let*

$$f\colon U \to \mathbb{C}$$

be a holomorphic function. Then

$$\forall\, z_0 \in U\ \forall\, \delta\ \exists\, z \in U\ \big(0 < |z - z_0| < \delta \wedge |f(z)| \geq |f(z_0)|\big)\ .$$

So for any holomorphic function f, the function $|f|$ has no strict local maximum. We will not prove it in these lectures.

Exercise 15 *Show that the function $f(x) := 1 - x^2$ disproves the maximum modulus principle for real functions.*

• *Segments and rectangles.* For proofs of Theorems 6 and 8 we need integrals over (straight) segments and over boundaries of rectangles. We define these geometric objects. For two different points $a, b \in \mathbb{C}$, the *segment* $u = ab \subset \mathbb{C}$ (between a and b) is the image

$$u = ab := \varphi[\,[0,\ 1]\,] = \{\varphi(t) \mid 0 \le t \le 1\} \subset \mathbb{C}$$

of the interval $[0, 1]$ in the linear function

$$\varphi(t) := (b - a)t + a\colon [0,\ 1] \to \mathbb{C}\ .$$

It is oriented by the order of its ends, so ab and ba are two different segments. It has *length* $|u| = |ab| := |b - a| \ge 0$. *The partition p of the segment $u = ab$* is a $(k + 1)$-tuple $p = (a_0, a_1, \ldots, a_k) \subset u$, $k \in \mathbb{N}$, of the points

$$a_i := \varphi(t_i),\ \ i = 0,\ 1,\ \ldots,\ k\ ,$$

which are images of the points $0 = t_0 < t_1 < \cdots < t_k = 1$ forming a partition of the interval $[0, 1]$. So $a_0 = a$, $a_k = b$ and the points $a_0, a_1, \ldots, a_k$ run on u from a to b. The *norm* $\|p\|$ of the partition p is

$$\|p\| := \max_{1 \le i \le k} |a_{i-1}a_i| = \max_{1 \le i \le k} |a_i - a_{i-1}|\ ,$$

i.e., it is the maximum length of a subsegment in the partition.

Exercise 16 *Prove that for each partition $p = (a_0, a_1, \ldots, a_k)$ of a segment $u = ab$,*

$$\sum_{i=1}^{k} |a_{i-1}a_i| = |ab|.$$

For a function $f\colon u \to \mathbb{C}$ and a partition $p = (a_0, a_1, \ldots, a_k)$ of the segment u we define the *Cauchy sum* $C(f,p)$, resp. its *modification* $C'(f,p)$, as

$$\begin{aligned} C(f,\,p) &:= \sum_{i=1}^{k} f(a_i)\cdot(a_i - a_{i-1}) \in \mathbb{C},\ \text{resp.} \\ C'(f,\,p) &:= \sum_{i=1}^{k} f(a_{i-1})\cdot(a_i - a_{i-1}) \in \mathbb{C}\ . \end{aligned}$$

They resemble Riemann sums from the definition of the Riemann integral.

Exercise 17 *Prove these estimates for the Cauchy sum and its modification:*

$$|C(f,\,p)|,\ |C'(f,\,p)| \le \sup_{z\in u} |f(z)|\cdot|u|\ .$$

A *rectangle* $R \subset \mathbb{C}$ is a set

$$R := \{z \in \mathbb{C} \mid \alpha \le \mathrm{re}(z) \le \beta \wedge \gamma \le \mathrm{im}(z) \le \delta\}$$

determined by real numbers $\alpha < \beta$ and $\gamma < \delta$. Its sides are parallel to the real and imaginary axis. When $\beta - \alpha = \delta - \gamma$, it is a *square*. *Canonical vertices* of the rectangle R are (a,b,c,d) in $\mathbb{C}^4$, where

$$a := \alpha + \gamma i,\ \ b := \beta + \gamma i,\ \ c := \beta + \delta i\ \text{ and }\ d := \alpha + \delta i.$$

They start at the bottom left corner and go counter-clockwise. *The boundary* ∂R of the rectangle R is the union of segments

$$\partial R := ab \cup bc \cup cd \cup da\ .$$

The *interior* $\mathrm{int}(R)$ of the rectangle R is

$$\mathrm{int}(R) := R \setminus \partial R\ .$$

The *perimeter* $\mathrm{per}(R)$ of the rectangle R is the sum of the lengths of its sides,

$$\mathrm{per}(R) := |ab| + |bc| + |cd| + |da| \ .$$

• *Integrals.* Let $f\colon u,\ \partial R \to \mathbb{C}$ be a continuous function defined on the segment u or on the boundary of a rectangle R. We define

$$\int_u f := \lim_{n\to\infty} C(f,\ p_n) \in \mathbb{C}$$

and

$$\int_{\partial R} f := \int_{ab} f + \int_{bc} f + \int_{cd} f + \int_{da} f \ ,$$

where (p_n) is any sequence of partitions p_n of the segment u such that $\lim \|p_n\| = 0$, and (a, b, c, d) are the canonical vertices of the rectangle R. The value of $\int_u f$ is the *integral of the function f over the segment u* and $\int_{\partial R} f$ is the *integral of the function f over the boundary of the rectangle R*. We prove correctness of these definitions and basic properties of these integrals.

Theorem 18 (on integrals) *Let $u = ab$, R and the functions f, g be as in the definitions above. The limit defining $\int_u f$ always exists and does not depend on the sequence (p_n). Thus also $\int_{\partial R} f$ is always correctly defined. Both integrals have the following properties.*

1. *For each $\alpha, \beta \in \mathbb{C}$, $\int_u(\alpha f + \beta g) = \alpha \int_u f + \beta \int_u g$ and the same holds for $\int_{\partial R}$.*
2. ML *estimates hold,*

$$\left|\int_u f\right| \le \max_{z\in u} |f(z)| \cdot |u| \quad \textit{and} \quad \left|\int_{\partial R} f\right| \le \max_{z\in\partial R} |f(z)| \cdot \mathrm{per}(R)$$

(see also Exercise 19).

3. *For each interior point c of the segment $u = ab$, that is $c \in ab$ and $c \neq a, b$, $\int_{ab} f = \int_{ac} f + \int_{cb} f$. Also, $\int_{ba} f = -\int_{ab} f$.*

Proof. Let $f\colon u \to \mathbb{C}$ be a continuous function. By Exercise 20, it suffices to prove that $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that for every two partitions p an q of the segment u with $\|p\|, \|q\| < \delta$ one has that

$$|C(f,\, p) - C(f,\, q)| < \varepsilon\ .$$

We show that this Cauchy condition for the partitions p and q is satisfied due to the uniform continuity of the function f. It follows from the continuity of f and compactness of the segment u (Exercise 21). For the proof of the Cauchy condition, first for a given $\varepsilon > 0$ we take a $\delta > 0$ that

$$x,\, y \in u \wedge |x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{|u|}\ .$$

Let $p = (a_0, a_1, \dots, a_k)$ and $q = (b_0, b_1, \dots, b_l)$ be two partitions of the segment u with $\|p\|, \|q\| < \delta$. We additionally assume that p refines q: $q \subset p$, hence $b_j = a_{i_j}$, $j = 0, 1, \dots, l$, for some indices $0 = i_0 < i_1 < \dots < i_l = k$. Then

$$C(f,\, p) \overset{(1)}{=} \sum_{j=1}^{l} C(f,\, p_j)\ ,$$

where $p_j := (a_{i_{j-1}}, a_{i_{j-1}+1}, \dots, a_{i_j})$ is the partition of the segment $u_j := a_{i_{j-1}} a_{i_j} = b_{j-1} b_j$, and

$$C(f,\, q) \overset{(2)}{=} \sum_{j=1}^{l} C(g_j,\, p_j)\ ,$$

where $g_j\colon u_j \to \mathbb{C}$ denotes a function that has the constant value $f(b_j)$ $(= f(a_{i_j}))$ on u_j. Then

$$
\begin{aligned}
& |C(f,\,q) - C(f,\,p)| \\
\overset{\text{eqs. (1) and (2), }\Delta\text{-ineq.}}{\leq}\ & \sum_{j=1}^{l} |C(g_j,\,p_j) - C(f,\,p_j)| \\
\overset{\text{def. of } p_j \text{ and } g_j}{\leq}\ & \sum_{j=1}^{l} \left| \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} (f(a_{i_j}) - f(a_m)) \cdot (a_m - a_{m-1}) \right| \\
\overset{\Delta \text{ ineq., } \delta \text{ and } a_m}{<}\ & \sum_{j=1}^{l} \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} \frac{\varepsilon}{|u|} \cdot |a_m - a_{m-1}| \\
\overset{\text{Exercise 16}}{=}\ & \sum_{j=1}^{l} \frac{\varepsilon}{|u|} \cdot |b_j - b_{j-1}| \overset{\text{Exercise 16}}{=} \frac{\varepsilon}{|u|} \cdot |u| = \varepsilon\ .
\end{aligned}
$$

For two general partitions we use the refinement trick. For a given $\varepsilon > 0$ we take the $\delta > 0$ whose existence we proved in the previous paragraph, i.e., such that for every two partitions p' and q' of the segment u, where $\|p'\|, \|q'\| < \delta$ and one of them refines the other, it holds that $|C(f,\,p') - C(f, q')| < \frac{\varepsilon}{2}$. Now if p and q are two arbitrary partitions of the segment u with $\|p\|, \|q\| < \delta$, we take their common refinement, the partition $r = p \cup q$. It refines both p and q and satisfies that $\|r\| < \delta$. By the definition of δ, we have the desired inequality:

$$
\begin{aligned}
|C(f,\,p) - C(f,\,q)| \ &\leq\ |C(f,\,p) - C(f,\,r)| + \\
&+\ |C(f, r) - C(f, q)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .
\end{aligned}
$$

This proves the existence of the integrals $\int_u f$ and $\int_{\partial R} f$.

The proofs of parts 1–3 by the limit transition $n \to \infty$ are easy. Part 1 follows from the linearity of the sum $C(\cdot, p_n)$ in the first variable. The first ML estimate follows from the estimation of Cauchy sums in the Exercise 17, and the second one follows from the first one. The first identity in part 3 follows from the additivity of the Cauchy sum in the second variable: $C(f,p) = C(f,q)+C(f,r)$, when q and r are partitions of the respective segments ac and cb, and $p = qr$ is the merged partition of the segment ab (clearly, $\|p\| = \max(\|q\|, \|r\|)$). The second identity follows from the fact that $\forall\, \varepsilon > 0\; \exists\, \delta > 0$ such that for each partition p of the segment u with $\|p\| < \delta$ one has that $|C(f,p) - C'(f,p)| < \varepsilon$ (a consequence of the uniform continuity of the function f) and from the fact that for each partition $p = (a_0, a_1, \ldots, a_k)$ of the segment ba one has that

$$\begin{aligned} C(f,\, p) \;&=\; \sum_{i=1}^{k} f(a_i)(a_i - a_{i-1}) = -\sum_{i=1}^{k} f(a_i)(a_{i-1} - a_i) \\ &=\; -C'(f,\, p')\;, \end{aligned}$$

where $p' = (a_0', a_1', \ldots, a_k') := (a_k, a_{k-1}, \ldots, a_0)$ is the partition of the segment ab obtained from p by reversing the segment ba. Of course, $\|p\| = \|p'\|$. □

Exercise 19 *Explain why the maxima in the second part of the previous theorem exist.*

Exercise 20 *Show that if the Cauchy condition holds for the Cauchy sums corresponding to the partitions p and q, then the finite limit defining $\int_u f$ exists and does not depend on the sequence (p_n).*

Exercise 21 *Let $A \subset M$ be a compact set in a metric space (M, d) and let $f\colon A \to N$ be a continuous function to the metric space (N, e). Prove that then f is uniformly continuous, that is,*

$$\forall\, \varepsilon\, \exists\, \delta \left(a,\, b \in A \wedge d(a,\, b) < \delta \Rightarrow e(f(a),\, f(b)) < \varepsilon\right)\, .$$

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 2, 7, 10, 16 a 21.