MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2023/24
lecturer: Martin Klazar

LECTURE 7 (April 2, 2024) SOLVING THE BASEL PROBLEM BY FOURIER SERIES

• *The Basel problem.* What is the sum of the series

$$B=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\dots\,?$$

According to Wikipedia (English mutation), this problem was presented by Pietro Mengoli in 1650 and solved by Leonard Euler in 1734:

$$B=\frac{\pi^2}{6}\,.$$

The problem is named after Euler's hometown. It was the residence of the mathematical clan of Bernoulli's who also tried to solve the problem but did not succeed.

• *Series.* We review basic notions of the theory of (infinite) series so that the previous problem makes sense. A *series* $\sum a_n=\sum_{n=1}^{\infty}a_n$ is in fact a sequence $(a_n)\subset\mathbb{R}$, to which we assign the sequence of *partial sums*

$$(s_n)=(a_1+a_2+\dots+a_n)\subset\mathbb{R}\,.$$

The limit of (s_n) is the *sum* of the series. If this limit is finite ($\in\mathbb{R}$), the series *converges*, else (the sum is $\pm\infty$ or does not exist) it *diverges*. The sum of a series is denoted by the same symbol as the series itself, so also

$$\sum a_n = \sum_{n=1}^{\infty} a_n = \lim\, s_n = \lim\left(a_1 + a_2 + \cdots + a_n\right) .$$

In exercises we review a few basic results about series.

Exercise 1 (necessary condition for convergence) *If the series* $\sum a_n$ *converges then* $\lim\, a_n = 0$*.*

Exercise 2 *If the series* $\sum a_n$ *has almost all summands non-negative, i.e.* $n \geq n_0 \Rightarrow a_n \geq 0$*, then* $\sum a_n$ *converges or has the sum* $+\infty$*.*

Exercise 3 (harmonic series) $\sum \frac{1}{n} = +\infty$.

Exercise 4 $\sum \frac{1}{(n+1)n} = 1$.

Exercise 5 *Using the previous problem, prove that the series* $\sum 1/n^2$ *in the Basel problem converges.*

Exercise 6 (geometric series) *For each* $q \in (-1, 1)$*,*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} .$$

Exercise 7 (the Leibniz Criterion) *When* $a_1 \geq a_2 \geq \cdots \geq 0$ *and* $\lim\, a_n = 0$*, then the series* $\sum(-1)^{n-1}a_n = a_1 - a_2 + a_3 - \ldots$ *converges.*

Exercise 8 *Derive in a simple way:*

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12} .$$

• *Riemannian series.* A series $\sum a_n$ is *Riemannian* if (i) $\lim a_n = 0$, (ii) $\sum a_{k_n} = +\infty$ and (iii) $\sum a_{z_n} = -\infty$, where (a_{k_n}), resp. (a_{z_n}), is the subsequence of nonnegative, resp. negative, summands in (a_n).

Exercise 9 (harder) *Fill in details in the sketch in the next proof.*

Theorem 10 (Riemann) *Let $\sum a_n$ be a Riemannian series. Then for every $S \in \mathbb{R}^*$ there is a permutation (bijection) $\pi\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S \ .$$

Thus by reordering any Riemannian series we can get any sum.

Proof. Suppose that $\sum a_n$ is a Riemannian series and that $\sum a_{k_n}$ and $\sum a_{z_n}$ are as in the definition. We define π for any given $S \in \mathbb{R}$ (i.e., S is a real number, not $\pm\infty$) as follows. We initialize three variables by $i = 1$, $j = 0$ and $\pi(1) = k_1$. Suppose that $\pi(1)$, $\pi(2)$, $\dots$, $\pi(n)$ have been already defined and $a = \sum_{k=1}^{n} a_{\pi(k)}$. If $a < S$ then $i := i + 1$, $j := j$ and $\pi(n + 1) = k_i$. If $a \ge S$ then $i := i$, $j := j+1$ and $\pi(n+1) = z_j$. In this way we define a map $\pi\colon \mathbb{N} \to \mathbb{N}$. It follows that π is a bijection and

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S \ .$$

$\square$

• *Trigonometric series.* These are the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \ ,$$

where $a_n, b_n \in \mathbb{R}$ are *coefficients* and $x \in \mathbb{R}$ is a variable. Effectively it is a parametric system of series parameterized by the variable x. Our goal is to derive expressions for a wide class of functions $f\colon [-\pi, \pi] \to \mathbb{R}$ as trigonometric series. Then we use it to solve the Basel problem.

Let $\mathcal{R}(-\pi, \pi)$ be the set of all Riemann integrable functions $f\colon [-\pi, \pi] \to \mathbb{R}$. For $f, g \in \mathcal{R}(-\pi, \pi)$ we define

$$\langle f,\, g\rangle = \int_{-\pi}^{\pi} fg \in \mathbb{R}$$

(it follows from the theory of the Riemann integral that if $f, g \in \mathcal{R}(-\pi, \pi)$, then $fg \in \mathcal{R}(-\pi, \pi)$ too). It looks like a scalar product:

Exercise 11 *Prove that*

$$\langle f,\, g\rangle = \langle g,\, f\rangle\,,\ \langle f,\, f\rangle \geq 0$$

and, for $a, b \in \mathbb{R}$*,*

$$\langle af + bg,\, h\rangle = a\,\langle f,\, h\rangle + b\,\langle g,\, h\rangle\ .$$

But it is not completely a scalar product:

Exercise 12 *The equivalence*

$$\langle f,\, f\rangle = 0 \iff f \equiv 0$$

does not hold.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is 2π-*periodic* if for every $x \in \mathbb{R}$ one has that $f(x + 2\pi) = f(x)$.

Proposition 13 (orthogonality of sines and cosines) *For every two integers* $m, n \geq 0$*,*

$$\langle \sin(mx),\, \cos(nx)\rangle = 0\ .$$

For every two integers $m, n \geq 0$, except $m = n = 0$, one has that

$$\langle \sin(mx), \sin(nx) \rangle = \langle \cos(mx), \cos(nx) \rangle = \begin{cases} \pi & \dots \quad m = n \quad \textit{and} \\ 0 & \dots \quad m \neq n \ . \end{cases}$$

Finally,

$$\langle \sin(0x), \sin(0x) \rangle = 0 \quad \textit{and} \quad \langle \cos(0x), \cos(0x) \rangle = 2\pi \ .$$

Proof. Let $m, n \in \mathbb{N}_0$. We compute the values

$$S_{m,n} = \langle \sin(mx), \sin(nx) \rangle, \ T_{m,n} = \langle \cos(mx), \cos(nx) \rangle$$

and

$$U_{m,n} = \langle \sin(mx), \cos(nx) \rangle \ .$$

Clearly, $S_{0,0} = 0$, $T_{0,0} = 2\pi$ and $U_{0,0} = 0$. Let m or n be non-zero, say $m \neq 0$ (for $n \neq 0$ the calculation is similar). Integration by parts using that $\sin(mx) = (-\cos(mx)/m)'$ and $\cos(mx) = (\sin(mx)/m)'$ yields

$$S_{m,n} = \frac{n}{m} \cdot T_{m,n}, \ T_{m,n} = \frac{n}{m} \cdot S_{m,n} \ \text{ and } \ U_{m,n} = -\frac{n}{m} \cdot U_{n,m}$$

– the first term $[\dots]_{-\pi}^{\pi}$ in the formula is always zero because ... is a 2π-periodic function. The first two equations together give

$$(1 - (n/m)^2) S_{m,n} = 0 = (1 - (n/m)^2) T_{m,n} \ .$$

If $n \neq m$ then $S_{m,n} = T_{m,n} = 0$. When $n = m$, then we know that $S_{m,m} = T_{m,m}$. But from the identity $\sin^2 x + \cos^2 x = 1$ (holding for every $x \in \mathbb{R}$) it follows that $S_{m,m} + T_{m,m} = \int_{-\pi}^{\pi} 1 = 2\pi$. Thus, $S_{m,m} = T_{m,m} = \pi$. The third equation above for $m = n$ gives $U_{m,m} = -U_{m,m}$ and so $U_{m,m} = 0$. To calculate $U_{m,n}$ for $m \neq n$, we express $U_{n,m}$ by integration by parts again using $\cos(mx) = (\sin(mx)/m)'$:

$$U_{n,m} = -(n/m)U_{m,n} \ .$$

Together $U_{m,n} = (n/m)^2 U_{m,n}$ and again $U_{m,n} = 0$. In summary: $S_{m,m} = T_{m,m} = \pi$ for $m \in \mathbb{N}$, $S_{0,0} = 0$ and $T_{0,0} = 2\pi$, and all other values of $S_{m,n}$, $T_{m,n}$ and $U_{m,n}$ for $m, n \in \mathbb{N}_0$ are zero. □

• *The Fourier series of a function.* For any function $f \in \mathcal{R}(-\pi, \pi)$ we define its *cosine Fourier coefficients*

$$a_n = \frac{\langle f(x),\, \cos(nx)\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{dx},\ \ n = 0,\,1,\,\ldots$$

and its *sine Fourier coefficients*

$$b_n = \frac{\langle f(x),\, \sin(nx)\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,\mathrm{dx},\ \ n = 1,\,2,\,\ldots\ .$$

The *Fourier series of the function* f ($\in \mathcal{R}(-\pi, \pi)$) is the trigonometric series

$$F_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big)\ ,$$

where a_n and b_n are, respectively, its cosine and sine Fourier coefficients. Geometrically viewed, we work in an infinite-dimensional vector space with the (almost) scalar product $\langle\cdot,\cdot\rangle$, in which the "coordinate axes" (elements of the orthogonal basis) are the functions

$$\{\cos(nx) \mid n \in \mathbb{N}_0\} \cup \{\sin(nx) \mid n \in \mathbb{N}\}\ .$$

Fourier coefficients of a given function f are its coordinates on these infinitely many coordinate axes. In contrast with Cartesian coordinates of points in $\mathbb{R}^n$, not every function is equal to the sum of its Fourier series. In a moment we present sufficient conditions (in Dirichlet's theorem and its corollary) for this to hold.

- *Bessel's inequality.*

Theorem 14 (Bessel's Inequality) *Fourier coefficients* a_n *and* b_n *of any function* $f \in \mathcal{R}(-\pi,\pi)$ *satisfy the inequality*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{\langle f,\, f\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 \,.$$

Proof. We denote by $s_n = s_n(x)$, $n = 1, 2, \ldots$, the n-th partial sum of the Fourier series of the function f:

$$\begin{aligned} s_n &= \frac{a_0}{2} + \sum_{k=1}^{n}\big(a_k \cos(kx) + b_k \sin(kx)\big) \\ &= \sum_{k=0}^{n}\big(a'_k \cos(kx) + b'_k \sin(kx)\big) \,, \end{aligned}$$

where

$$a_k = \pi^{-1}\langle f,\, \cos(kx)\rangle,\ b_k = \pi^{-1}\langle f,\, \sin(kx)\rangle,\ k = 0,\, 1,\, 2,\, \ldots\ ,$$

$a'_0 = a_0/2$, $a'_k = a_k$ for $k > 0$, $b'_0 = 0$ and $b'_k = b_k$ for $k > 0$. Due to the linearity of the (almost) scalar product $\langle\cdot,\cdot\rangle$, definition of numbers a'_k, b'_k, a_k, b_k and orthogonality of functions $\sin(kx)$ and $\cos(kx)$ we have

$$\begin{aligned} \langle s_n,\, s_n\rangle &= \sum_{k=0}^{n}\big((a'_k)^2\langle\cos(kx),\, \cos(kx)\rangle + (b'_k)^2\langle\sin(kx),\, \sin(kx)\rangle\big) \\ &= \pi\left(\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)\right) \end{aligned}$$

and also

$$\begin{aligned}\langle s_n,\, f\rangle &= \sum_{k=0}^{n}\left(a'_k\langle\cos(kx),\, f\rangle + b'_k\langle\sin(kx),\, f\rangle\right)\\ &= \pi\left(\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2)\right)\ .\end{aligned}$$

On the other hand,

$$0 \le \langle f - s_n,\, f - s_n\rangle = \langle f,\, f\rangle - 2\langle s_n,\, f\rangle + \langle s_n,\, s_n\rangle\ ,$$

hence $2\langle s_n, f\rangle - \langle s_n, s_n\rangle \le \langle f, f\rangle$. Thus for every n,

$$\frac{a_0^2}{2}+\sum_{k=1}^{n}(a_k^2+b_k^2) = \frac{2\langle s_n, f\rangle - \langle s_n, s_n\rangle}{\pi} \le \frac{\langle f, f\rangle}{\pi}\ .$$

The series of squares of the Fourier coefficients of the function f converges and its sum is bounded by the stated value. □

Exercise 15 (Riemann–Lebesgue Lemma) *Using Bessel's inequality, prove that for every function $f \in \mathcal{R}(-\pi, \pi)$*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{dx} = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{dx} = 0\ .$$

(Hint: see Exercise 1).

• *Piece-wise smooth functions and Dirichlet's theorem.* The function

$$f\colon [a,\, b] \to \mathbb{R}\ ,$$

where $a < b$ are real numbers, is *piece-wise smooth* if there is a partition

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b,\ \ k \in \mathbb{N}\ ,$$

of the interval $[a, b]$ such that on every interval (a_{i-1}, a_i), $i = 1, 2, \ldots, k$, f has continuous derivative f', for every $i = 1, 2, \ldots, k$ there exist finite one-sided limits

$$f(a_i - 0) = \lim_{x \to a_i^-} f(x) \quad \text{and} \quad f'(a_i - 0) = \lim_{x \to a_i^-} f'(x)$$

and for each $i = 0, 1, \ldots, k-1$ there exist finite one-sided limits

$$f(a_i + 0) = \lim_{x \to a_i^+} f(x) \quad \text{and} \quad f'(a_i + 0) = \lim_{x \to a_i^+} f'(x)\ .$$

A piece-wise smooth function can be at several points in the interval $[a, b]$ discontinuous, but at the points of discontinuity it has finite one-sided limits and one-sided non-vertical tangents.

Exercise 16 *Is the function* $f\colon [-1, 1] \to \mathbb{R}$*, defined as* $f(x) = (-x)^{1/3}$ *for* $x \in [-1, 0]$ *and* $f(x) = x^{1/3}$ *for* $x \in [0, 1]$*, piece-wise smooth?*

Exercise 17 *Is the signum function* $\operatorname{sgn}\colon [-1, 1] \to \mathbb{R}$*, defined as* $\operatorname{sgn}(x) = -1$ *for* $x \in [-1, 0)$, $\operatorname{sgn}(0) = 0$ *and* $\operatorname{sgn}(x) = 1$ *for* $x \in (0, 1]$*, piece-wise smooth?*

Theorem 18 (Dirichlet's) *Let*

$$f\colon \mathbb{R} \to \mathbb{R}$$

be a 2π*-periodic function such that its restriction to the interval* $[-\pi, \pi]$ *is piece-wise smooth. Then for every* $a \in \mathbb{R}$ *its Fourier series* $F_f(x)$ *sums to*

$$F_f(a) = \frac{f(a+0) + f(a-0)}{2} = \frac{\lim_{x \to a^+} f(x) + \lim_{x \to a^-} f(x)}{2}\ .$$

Thus, at each point of continuity $a \in \mathbb{R}$ *of the function* $f(x)$*, its Fourier series sums to the functional value,* $F_f(a) = f(a)$*.*

Proof. We will probably skip it. □

We say that the function $f\colon [a,b] \to \mathbb{R}$ is *smooth* if it has on (a,b) continuous derivative f' and at the ends a and b the functions $f(x)$ and $f'(x)$ have finite limits.

Corollary 19 (on smooth function) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic and continuous function whose restriction to the interval $[-\pi,\pi]$ is smooth. Then for each $a \in \mathbb{R}$ is*

$$F_f(a) = f(a) \ .$$

Any continuous and smooth function is therefore equal to the sum of its Fourier series.

Proof. This follows from the previous theorem: by the assumption f is continuous on $\mathbb{R}$. □

• *Back to the Basel problem.* Let $I \subset \mathbb{R}$ be an interval symmetric with respect to the origin and $f\colon I \to \mathbb{R}$. We say that the function f is *even* (resp. *odd*) if for every $x \in I$, $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$).

Exercise 20 *Let $f \in \mathcal{R}(-\pi,\pi)$. Prove that all sine (or cosine) Fourier coefficients of an even (or odd) functions f are zero. How do you simplify cosine (or sine) Fourier coefficients of an even (or odd) function?*

We calculate the Fourier series of the function $f\colon \mathbb{R} \to \mathbb{R}$ defined on the interval $[-\pi,\pi]$ by $f(x) = x^2$ and 2π-periodically extended to the entire $\mathbb{R}$ (which is possible due to the fact that $(-\pi)^2 = \pi^2$). Its sine Fourier coefficients are zero according to the previous exercise. The first (actually zero) cosine Fourier coefficient is (according to this exercise)

$$a_0 = \frac{2}{\pi}\int_0^\pi x^2 \,\mathrm{dx} = \frac{2\pi^2}{3} \,.$$

Next ($n \in \mathbb{N}$)

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x^2 \overbrace{\cos(nx)}^{(\sin(nx)/n)'} \,\mathrm{dx} \\
&= \frac{2}{\pi n}\underbrace{[x^2\sin(nx)]_0^\pi}_{0-0=0} - \frac{4}{\pi n}\int_0^\pi x \overbrace{\sin(nx)}^{(-\cos(nx)/n)'} \,\mathrm{dx} \\
&= \frac{4}{\pi n^2}\underbrace{[x\cos(nx)]_0^\pi}_{\pi(-1)^n} - \frac{4}{\pi n^2}\underbrace{\int_0^\pi \cos(nx)\,\mathrm{dx}}_{0-0=0} \\
&= (-1)^n\frac{4}{n^2} \,.
\end{aligned}$$

Since the function f is continuous and smooth on $[-\pi,\pi]$, by Corollary 19 one has for every $a \in \mathbb{R}$ that

$$f(a) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(na) = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty (-1)^n\frac{\cos(na)}{n^2} \,.$$

For $a = \pi$ we get

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4\sum_{n=1}^\infty (-1)^n\frac{(-1)^n}{n^2}, \quad \text{so} \quad \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} \,.$$

Exercise 21 *The function $f(x)$ is defined on the interval $[-\pi,\pi)$ as $f(x) = \pi - x$ and is 2π-periodically extended to $\mathbb{R}$. Expand it into Fourier series.*

Exercise 22 *What sum of the infinite series do we get from the previous expansion (using Dirichlet's theorem) for $x = \frac{\pi}{2}$?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 8, 9, 16 and 20.