MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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LECTURE 6 (March 26, 2024)

APPLICATIONS OF BAIRE'S THEOREM: NON-DIFFERENTIABLE CONTINUOUS FUNCTIONS, TRANSCENDENTAL GROWTHS OF PERMUTATIONS

• *Non-differentiable continuous functions.* Let $I = [0, 1]$. By $C(I)$ we denote the set of all continuous functions from I to $\mathbb{R}$. Recall that for $x \in \mathbb{R}$ and $\delta > 0$,

$$P(x,\, \delta) = (x - \delta,\, x + \delta) \setminus \{x\} = (x - \delta,\, x) \cup (x,\, x + \delta)$$

is the deleted δ-neighborhood of x. We prove the following theorem.

Theorem 1 (wild functions exist) *There exists a function f in $C(I)$ such that for every $x \in I$ and every $\delta > 0$,*

$$\sup\left(\left\{\left|\frac{f(y) - f(x)}{y - x}\right| \;\middle|\; y \in P(x,\, \delta) \cap I\right\}\right) = +\infty\ .$$

Recall that $f\colon I \to \mathbb{R}$ is differentiable at $x \in I$ if it has a finite derivative $f'(x) \in \mathbb{R}$.

Exercise 2 *The function f in Theorem 1 is continuous on I but is not differentiable at any point of I.*

• *Four lemmas.* We prove Theorem 1 with the help of four lemmas.

Lemma 3 (1st lemma) *If $f \in C(I)$ has the property that for every $x \in I$,*

$$\sup\left(\left\{\left|\frac{f(y) - f(x)}{y - x}\right| \;\middle|\; y \in I \setminus \{x\}\right\}\right) = +\infty$$

then f has the property in Theorem 1. Hence the parameter δ in Theorem 1 is superfluous.

Proof. We assume that $f \in C(I)$ has for every $x \in I$ the stated property. The set

$$Q(x, \delta) = I \setminus U(x, \delta) = [0, 1] \setminus (x - \delta, x + \delta)$$

is compact for every $x \in I$ and every $\delta > 0$ (Exercise 4). Let $M(x, \delta)$ be the maximum value of the continuous function

$$Q(x, \delta) \ni y \mapsto \big|(f(y) - f(x))/(y - x)\big| \geq 0 \, .$$

For every given $x \in I$ and $\delta > 0$, by the assumption there is a $y \in I \setminus \{x\}$ such that

$$\left| \frac{f(y) - f(x)}{y - x} \right| > M(x, \delta) \, .$$

But then $y \notin Q(x, \delta)$, thus $y \in P(x, \delta)$ and wee see that f has the property in Theorem 1. □

Exercise 4 *Show that the set $Q(x, \delta)$ is compact.*

Exercise 5 *Why is the function $y \mapsto \left|\frac{f(y)-f(x)}{y-x}\right|$ continuous?.*

Recall that for any set X, the infinity-norm

$$\|f\|_\infty = \sup(\{|f(x)| \mid x \in X\})$$

on the set B of bounded functions $f \colon X \to \mathbb{R}$ makes B a MS

$$(B, \|f - g\|_\infty).$$

Exercise 6 *Show that this is a* MS.

Lemma 7 (2nd lemma) *Let* (M, d) *be a* MS, $(x_n) \subset M$ *be a sequence with* $\lim x_n = x_0 \in M$ *and let* (f_n), $f_n\colon M \to \mathbb{R}$, *be a sequence of functions converging in the norm* $\|\cdot\|_\infty$ *to a continuous function* $f\colon M \to \mathbb{R}$. *Then*

$$\lim f_n(x_n) = f(x_0)\ .$$

Proof. By the triangle inequality,

$$|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|\ .$$

For a given $\varepsilon > 0$, we can make the first $|\cdot|$ on the right side $< \varepsilon/2$ for $n \ge n_0$ due to the assumption that $\|f_n - f\|_\infty \to 0$. The same holds for the second $|\cdot|$ on the right side if $n \ge n_1$ due to the Heine definition of continuity of f at the point x_0. Hence $n \ge \max(\{n_0, n_1\}) \Rightarrow |f_n(x_n) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

A *broken line* going through the points (a_0, b_0), (a_1, b_1), ..., (a_k, b_k) in $\mathbb{R}^2$ in this order, where $a_0 < a_1 < \cdots < a_k$, is the function $f\colon [a_0, a_k] \to \mathbb{R}$ which is on every interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, k$, defined by

$$f(x) = \frac{(b_i - b_{i-1})(x - a_{i-1})}{a_i - a_{i-1}} + b_{i-1}$$

(thus $f(a_{i-1}) = b_{i-1}$ and $f(a_i) = b_i$). Its graph on the interval $[a_{i-1}, a_i]$ is the segment joining the points (a_{i-1}, b_{i-1}) and (a_i, b_i). We call these segments just *segments*.

Exercise 8 *Every broken line is a continuous function.*

The *slope* of a plane line given by the equation $y = ax + b$ is the number a. The *slope* of a segment is the slope of the line extending the segment. The *secant (line)* of a function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, is a line going through two distinct points on the graph of f.

Lemma 9 (3rd lemma) *For every $\varepsilon > 0$ and every function $f \in C(I)$ there is a function $g \in C(I)$ and a constant $M > 0$ such that*

$$(i)\ \|f - g\|_\infty < \varepsilon \ \text{ and } \ (ii)\ x,\, y \in I,\, x \neq y \Rightarrow \left| \frac{g(y) - g(x)}{y - x} \right| < M$$

– every $f \in C(I)$ can be arbitrarily closely approximated by a $g \in C(I)$ whose secant lines have bounded slopes.

Proof. Let $f \in C(I)$ and let an $\varepsilon > 0$ be given. Since the interval I is compact, the function f is uniformly continuous (Exercise 10). Hence for every sufficiently large m and every $i = 0, 1, \ldots, m$ it holds that

$$\tfrac{i}{m} \le x \le \tfrac{i+1}{m} \Rightarrow |f(\tfrac{i}{m}) - f(x)|,\ |f(\tfrac{i+1}{m}) - f(x)| < \tfrac{\varepsilon}{2}\,.$$

We draw through the points $(i/m, f(i/m))$, $i = 0, 1, \ldots, m$ a broken line g. For g the above implication holds too and with the same m (Exercise 11). Thus

$$\forall\, x \in I\ \big(|f(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\big)$$

(Exercise 12) and g has property (i). By Exercise 13 we have that for every two distinct numbers $x, y \in I$,

$$\left| \frac{g(y) - g(x)}{y - x} \right| \le s$$

where s is the largest, in absolute value, slope of a segment of the broken line g. Hence g has also property (ii). $\square$

Exercise 10 *Why is any $f \in C(I)$ uniformly continuous?*

Exercise 11 *Show that the displayed implication holds for the broken line* g.

Exercise 12 *Prove the displayed inequality that* $\forall\, x \ldots$.

Exercise 13 *Prove the inequality* $\cdots \le s$.

Lemma 14 (4th lemma) *For every* $\varepsilon > 0$ *and* $T > 0$ *there is a function* $g \in C(I)$ *such that*

$$(i)\ \|g\|_\infty < \varepsilon\ \ \textit{and}\ \ (ii)\ \forall\, x \in I\, \exists\, y \in I \setminus \{x\} : \left|\frac{g(y) - g(x)}{y - x}\right| > T$$

– there is a continuous and $\|\cdot\|_\infty$*-small function* g*, defined on* I*, such that through every point of its graph goes a secant line with a large slope.*

Proof. Let $\varepsilon > 0$ and $T > 0$ be given. We take a large even $m \in \mathbb{N}$ such that $2m\varepsilon/3 > T$, take the $m + 1$ points

$$\big(i/m,\ (\varepsilon/3)(1 - (-1)^i\big) \in \mathbb{R}^2,\ i = 0,\, 1\, \ldots,\, m\ ,$$

in the plane and draw through them a broken line g. It joins $(0, 0)$ and $(1, 0)$ and has $m/2$ hills, each with height $2\varepsilon/3$ and base width $2/m$. Thus $\|g\|_\infty = 2\varepsilon/3 < \varepsilon$ and (i) holds. Let u be a point on the graph of g. We lead through it the secant line extending the segment containing u (if u lies in two segments, we choose any of them). Its |slope| $> T$ because both sides of any hill have |slope| $= \frac{2\varepsilon/3}{1/m} = \frac{2m\varepsilon}{3} > T$. We satisfied (ii) too. □

• *Proof of Theorem 1.* We show that there is a continuous function $f\colon I \to \mathbb{R}$ that is not differentiable at any point of I.

Proof of Theorem 1. For $n \in \mathbb{N}$ we define sets

$$A_n = \left\{ f \in C(I) \mid \exists\, x \in I\; \forall\, y \in I \setminus \{x\} \left(\left|\tfrac{f(y)-f(x)}{y-x}\right| \le n \right) \right\} .$$

We show that every set A_n is a sparse subset of the MS

$$(C(I),\ \|f-g\|_\infty)$$

and by this we will be done. Indeed, by Proposition 17 below this MS is complete and therefore by Baire's theorem there exists a function

$$f \in C(I) \setminus \bigcup_{n=1}^{\infty} A_n .$$

Thus f is continuous and has the property described in the first Lemma 3 and therefore, by this lemma, has the property in Theorem 1 and by Exercise 2 the function f is not differentiable at any point of I.

We show that every set $A_n \subset C(I)$ is closed and contains no ball, i.e., that for every ball $B(f,r)$ in the MS, $B(f,r) \not\subset A_n$. It follows from this that A_n is a sparse set (Exercise 15).

We prove that A_n is closed by showing its closedness to limits. Let $(f_k) \subset A_n$ be a sequence with $\lim_{k\to\infty} f_k = f \in C(I)$; we show that $f \in A_n$. Since $f_k \in A_n$, there is a number $x_k \in I$ such that for every $y \in I \setminus \{x_k\}$,

$$\left|\frac{f_k(y) - f_k(x_k)}{y - x_k}\right| \le n .$$

We know from *Mathematical Analysis* 1 that (x_k) has a convergent subsequence with a limit in I. To simplify notation, we assume that already $\lim_{k\to\infty} x_k = x_0 \in I$. For every $y \in I \setminus \{x_0\}$ we have, by the property of the point x_k and the second Lemma 7, that

$$n \geq \lim_{k\to\infty} \left|\frac{f_k(y) - f_k(x_k)}{y - x_k}\right| = \left|\frac{f(y) - f(x_0)}{y - x_0}\right|$$

(non-strict inequalities are preserved in limits). The number x_0 therefore witnesses that $f \in A_n$ and A_n is a closed subset of the MS.

It remains to find in the given ball $B(f, r) \subset C(I)$ a point (i.e., a function) $g \in B(f, r) \setminus A_n$. We define it as $g = g_1 + g_2$ where we get the functions g_1 and g_2 using the third and fourth Lemma 9 and 14, respectively. First we use Lemma 9 and get a function $g_1 \in C(I)$ and a constant $M > 0$ such that $\|f - g_1\|_\infty < r/2$ and that all secants of the graph of g_1 have slope in absolute value $< M$. Then we use Lemma 14 and get a function $g_2 \in C(I)$ such that $\|g_2\|_\infty < r/2$ and that through every point on the graph of g_2 there goes a secant line with slope in absolute value $> M + n$. By the triangle inequality,

$$\|f - g\|_\infty \leq \|f - g_1\|_\infty + \|g_2\|_\infty < r/2 + r/2 = r$$

and $g \in B(f, r)$. Let $x \in I$ be arbitrary. By the property of the function g_2 we take a $y \in I \setminus \{x\}$ such that $|\frac{g_2(y)-g_2(x)}{y-x}| > M + n$. Then

$$\begin{aligned} \left|\frac{g(y) - g(x)}{y - x}\right| &= \left|\frac{g_2(y) - g_2(x)}{y - x} + \frac{g_1(y) - g_1(x)}{y - x}\right| \\ &\geq \left|\frac{g_2(y) - g_2(x)}{y - x}\right| - \left|\frac{g_1(y) - g_1(x)}{y - x}\right| \\ &> (M + n) - M = n \end{aligned}$$

and $g \notin A_n$. On the first line we used the definition of g, on the second the inequality from Exercise 16 and on the third the properties of the functions g_1 and g_2. □

Exercise 15 *Prove that every closed set X (in a* MS*) with empty interior (i.e., X contains no ball) is sparse.*

Exercise 16 *Prove that for every two real numbers a and b,*

$$|a - b| \geq |a| - |b| \ .$$

• *Completeness of the* MS *of continuous functions with the infinity-norm metric.*

Proposition 17 *The metric space*

$$(C(I), \|f - g\|_\infty)$$

is complete.

Proof. Let $(f_n) \subset C(I)$ be a Cauchy sequence in this MS, i.e.,

$$\forall\, \varepsilon > 0\ \exists\, m\ \big(n,\, n' \geq m \Rightarrow \|f_n - f_{n'}\|_\infty < \varepsilon\big) \ .$$

Then for every $x \in I$ the sequence $(f_n(x)) \subset \mathbb{R}$ is Cauchy, therefore convergent, and we can define

$$f(x) = \lim f_n(x) \ .$$

Thus we have a function $f\colon I \to \mathbb{R}$ with the property that $f_n \to f$ pointwisely. Let us prove the uniform convergence, i.e., that $\|f - f_n\|_\infty \to 0$. Let an $x \in I$ and an $\varepsilon > 0$ be given. We take an m (it is independent of x) such that the above displayed Cauchy condition holds with $\varepsilon/2$. Then we take a $k \geq m$ such that $|f_k(x) - f(x)| < \varepsilon/2$. Thus $n \geq m \Rightarrow$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_k(x)| + |f_k(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and $\lim f_n = f$ in this MS.

It remains to show that f is continuous (i.e., it is an element of this MS). Let an $x_0 \in I$ and an $\varepsilon > 0$ be given. We take an n_0 such that

$$n \geq n_0 \Rightarrow \|f - f_n\|_\infty \leq \varepsilon/2 \ .$$

We take a $\delta > 0$ such that

$$x \in U(x_0,\, \delta) \cap I \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| \leq \varepsilon/2$$

(we use the continuity of f_{n_0} at x_0). Then $\forall\, x \in U(x_0, \delta) \cap I$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

— f is continuous at x_0. □

Now the proof of Theorem 1 is complete.

• *An application of Baire's theorem in enumeration of permutations.* For $m \leq n$ in $\mathbb{N} = \{1, 2, \ldots\}$ and two permutations (i.e., bijections) $\pi\colon [m] \to [m]$ and $\rho\colon [n] \to [n]$ we write $\pi \preceq \rho$, and say that π *is contained in* ρ, if there exist numbers $i_1 < i_2 < \cdots < i_m$ in $[n]$ such that

$$\forall\, j,\, k \in [m] \left(\pi(j) < \pi(k) \iff \rho(i_j) < \rho(i_k)\right) \ .$$

Let $\mathcal{S}$ be the set of all finite permutations $\pi\colon [n] \to [n]$ for n running in $\mathbb{N}$ and let $\mathcal{S}_n \subset \mathcal{S}$ be the ($n!$-element) set of permutations of $[n]$.

Exercise 18 *Show that $(\mathcal{S}, \preceq)$ is a non-strict partial order.*

We say that a set $X \subset \mathcal{S}$ is a *permutation class* if for every two permutations π and ρ,

$$\pi \preceq \rho \in X \Rightarrow \pi \in X \ .$$

In the last cca 20 years, many results on enumeration of permutation classes X, i.e., on the counting functions of the form

$$n \mapsto |X \cap \mathcal{S}_n|$$

($|A|$ denotes the cardinality of a finite set A), were obtained. A basic one is the next theorem.

Theorem 19 (A. Marcus and G. Tardos, 2004) *Let X be a permutation class. Then*

$$X \neq \mathcal{S} \Rightarrow \exists\, c > 1\, \forall\, n \left(|X \cap \mathcal{S}_n| \le c^n\right) .$$

In words, any permutation class, with the exception of the class of all permutations, grows only at most exponentially.

Exercise 20 *Let $\pi \in \mathcal{S}_2$ be the identical permutation ($\pi(1) = 1$, $\pi(2) = 2$) and let X be any permutation class such that $\pi \notin X$. Show that then $|X \cap \mathcal{S}_n| \le 1$ for every n.*

By the Marcus–Tardos theorem, for every permutation class X different from $\mathcal{S}$ one can define its finite *growth rate*

$$c(X) = \limsup_{n \to \infty} |X \cap \mathcal{S}_n|^{1/n} \in [0, +\infty) .$$

For example, it is known that $c(\{\rho \in \mathcal{S} \mid \rho \not\succeq \pi\}) = 4$ for any $\pi \in \mathcal{S}_3$. In fact,

$$|X \cap \mathcal{S}_n| = \frac{1}{n+1}\binom{2n}{n}$$

for every n for any of these six permutations classes X.

For some time there was a conjecture that every growth rate of a permutation class is an algebraic number. It was refuted by the following result.

Theorem 21 (M. Albert and S. Linton, 2009) $\exists$ *a set*

$$A \subset [0, +\infty)$$

such that $A \neq \emptyset$, is closed, has no isolated point and every $x \in A$ is the growth rate of a permutation class.

As we saw in the lecture before the last lecture, Baire's theorem implies that each such set A is uncountable. Thus we have uncountably many growth rates of permutation classes, and (since the set of algebraic numbers is countable) almost all of them are non-algebraic.

Corollary 22 (transcendental growths) *Hence there exist non-algebraic growth rates of permutation classes.*

Exercise 23 *How does it exactly follow from Baire's theorem that the above set A is uncountable?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 2, 4, 15, 20 and 23.