MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
lecturer: Martin Klazar

LECTURE 5 (March 19, 2024) THE WEAK JORDAN CIRCUIT THEOREM AFTER THOMASSEN

I show you a proof modeled after the remarkable proof *Carsten Thomassen (1948)* gave in 1992 for the famous theorem of *Camille Jordan (1838–1922)* in topology of the plane. We prove that any topological circuit disconnects $\mathbb{R}^2$. We will follow the article The Jordan Curve Theorem, Formally and Informally, *Amer. Math. Monthly* **114** (2007), 882–894 by *Thomas C. Hales (1958)*.

Exercise 1 *Learn about achievements of these three excellent mathematicians.*

• *Arcs, circuits,* PL *maps and two theorems.* Let $I = I_{a,b} = [a, b]$ where $a < b$ are in $\mathbb{R}$. An *arc* is an injective continuous map

$$f\colon I \to \mathbb{R}^2\,.$$

Here I is a subspace of the MS $(\mathbb{R}, e_1)$ and $\mathbb{R}^2$ is the MS $(\mathbb{R}^2, e_2)$. The points $f(a)$ and $f(b)$ in $\mathbb{R}^2$ are the *endpoints of f*. We say that f *joins $f(a)$ to $f(b)$*. If $f[I] \subset X \subset \mathbb{R}^2$, we say that f *joins $f(a)$ to $f(b)$ in X*. The set $f^0 = f[I]^0 = f[(a, b)]$ is the *interior (of the arc f)*. The *distance between two sets $A, B \subset \mathbb{R}^2$* is the infimum

$$e_2(A,\, B) = \inf(\{e_2(\overline{x},\, \overline{y}) \mid \overline{x} \in A,\, \overline{y} \in B\})\ .$$

If A and B are disjoint and compact, their distance is positive.

A *circuit* is any continuous map $f\colon I = I_{a,b} \to \mathbb{R}^2$ such that $f(x) \ne f(y)$ for $x \ne y$, with the exception of $f(a) = f(b)$. We extend f to the $(b-a)$-periodic function $f_{\mathrm{e}}\colon \mathbb{R} \to \mathbb{R}^2$.

A map $f\colon I \to \mathbb{R}^2$ is PL (*piece-wise linear*) if there exist a partition $a = t_0 < t_1 < \dots < t_n = b$, $n \in \mathbb{N}$, of I and $2n$ vectors $\overline{s_i}, \overline{c_i} \in \mathbb{R}^2$, $i = 1, \dots, n$, such that no $\overline{s_i} = (0,0)$ and

$$\forall\, i \in [n]\ \forall\, t \in [t_{i-1},\, t_i]\ \big(f(t) = t \cdot \overline{s_i} + \overline{c_i}\big)\ .$$

The points $f(t_i) \in \mathbb{R}^2$, $i = 0, 1, \dots, n$, are the *corners (of f)*. The plane straight segments $f[\,[t_{i-1}, t_i]\,]$ are the *segments (of f)*. For any two points $\overline{p}, \overline{q} \in \mathbb{R}^2$ we denote by $s(\overline{p}, \overline{q})$ the straight segment joining them. An *axis* of a corner $f(t_i)$ with $0 < i < n$ is the line going through $f(t_i)$ that halves both angles at $f(t_i)$ (determined by the two segments of f incident with the corner). If $f(t_0) = f(t_n)$, this corner also has an axis. An *oriented* PL *circuit* is one where all segments are oriented consistently in one of the two ways.

Exercise 2 *Show that any* PL *map is continuous.*

Two famous theorems about images of arcs and circuits are

Theorem 3 (the Arc Theorem) *For any arc $f\colon I \to \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is connected.*

and (`https://kam.mff.cuni.cz/~klazar/JordanPic1.pdf`)

Theorem 4 (the Weak Jordan Theorem) *For any circuit $f\colon I \to \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is disconnected.*

We prove the latter theorem. We discuss the full Jordan Theorem in the concluding remarks. Not completely convincing outline of a proof of the Arc Theorem is given in [Hales], pp. 890–891.

Exercise 5 *In the two theorems, the complements are open plane sets.*

• *Two results on connected sets.* Recall that $X \subset \mathbb{R}^2$ is connected if $\neg\exists$ open (or closed) sets $A, B \subset \mathbb{R}^2$ cutting X, i.e., such that $X \subset A \cup B$, both intersections $X \cap A$ and $X \cap B$ are nonempty and $X \cap A \cap B = \emptyset$. A set $X \subset \mathbb{R}^2$ is PL-*connected* if for any $x, y \in X$, $x \neq y$, there is a PL arc $f\colon I \to \mathbb{R}^2$ joining x to y in X.

Theorem 6 (conn. $\leftrightarrow$ PL-conn.) *Any open set $X \subset \mathbb{R}^2$ is connected iff it is* PL-*connected.*

Proof. $\neg \Rightarrow \neg$. Suppose that A and B cut X, $x \in X \cap A$, $y \in X \cap B$ and that the PL arc $f\colon I \to \mathbb{R}^2$ with $f[I] \subset X$ joins x to y. Then A and B cut $f[I]$ and $f[I]$ is disconnected. This is impossible because $f[I]$ is connected as a continuous image of the connected interval I. Hence f does not exist.

$\neg \Leftarrow \neg$. Consider the partition $X/\sim$ of X by the equivalence relation $\sim$ on X (Exercise 8) defined by $x \sim y$ iff a PL arc joins x to y in X. It is not hard to see that every block $A \in X/\sim$ is an open set. We assume that X is not PL-connected: $|X/\sim| \geq 2$. We take any block $A \in X/\sim$ and define B to be the union of all other blocks. Then A and B are open sets cutting X. Thus X is disconnected. $\square$

Exercise 7 *Describe a set $X \subset \mathbb{R}^2$ that is a countable union of plane segments and is connected but is not* PL-*connected.*

Exercise 8 *Show that the relation $\sim$ in the previous proof is transitive. Or, better, prove this more general proposition.*

Proposition 9 (PL maps and PL arcs) *For every* PL *map* $f\colon I = I_{a,b} \to \mathbb{R}^2$ *there is a* PL *arc* $g\colon I' \to \mathbb{R}^2$ *such that* $g[I'] \subset f[I]$ *and* g *joins* $f(a)$ *to* $f(b)$.

• (PL) *configurations.* A *C-configuration*, abbreviated *C-conf*, is any circuit $f\colon I \to \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus f[I]$ is connected. Our goal is to prove that no C-conf exists, i.e., that Theorem 4 holds. A $K_{3,3}$-*configuration*, abbreviated $K_{3,3}$-*conf*, is any nine-tuple of arcs $f_{i,j}$, $i, j \in [3]$, such that their endpoints form a six-element set

$$K = \{\overline{p_1},\, \overline{p_2},\, \overline{p_3},\, \overline{q_1},\, \overline{q_2},\, \overline{q_3}\} \subset \mathbb{R}^2\ ,$$

for every pair $i, j \in [3]$ the arc $f_{i,j}$ joins $\overline{p_i}$ to $\overline{q_j}$, and the nine interiors $f^0_{i,j}$ are pairwise disjoint and disjoint to K. Graph-theoretically, a $K_{3,3}$-conf is a plane drawing (i.e., without crossings) of the complete bipartite graph $K_{3,3}$.

Exercise 10 *Explain why no* $K_{3,3}$-*conf exists. Hint: recall the course* Discrete Mathematics.

A PL *C-configuration*, abbreviated PL *C-conf*, is any C-conf in which the circuit f is a PL map. Similarly, a PL $K_{3,3}$-*configuration*, abbreviated PL $K_{3,3}$-*conf*, is any $K_{3,3}$-conf in which all nine arcs $f_{i,j}$ are PL maps. Now we start the proof of Theorem 4.

• *Thomassen's reductions.* The proof is split in four reductions.

1. $\exists$ C-conf $\Rightarrow$ $\exists$ $K_{3,3}$-conf
2. $\exists$ $K_{3,3}$-conf $\Rightarrow$ $\exists$ PL $K_{3,3}$-conf
3. $\exists$ PL $K_{3,3}$-conf $\Rightarrow$ $\exists$ PL C-conf
4. $\exists$ PL C-conf $\Rightarrow 0 = 1$

When we prove these four implications, Theorem 4 will follow. The main invention of Thomassen is the first reduction.

Exercise 11 *Exercise 10 says that* $\exists\ K_{3,3}$*-conf* $\Rightarrow 0 = 1$. *Does not this simplify our proof?*

• *The first reduction* $\exists\ C$*-conf* $\Rightarrow \exists\ K_{3,3}$*-conf.* See the picture `https://kam.mff.cuni.cz/~klazar/JordanPic2.pdf`
Let $f\colon I = I_{a,b} \to \mathbb{R}^2$ be a C-conf, i.e., f is a circuit such that the open set $\mathbb{R}^2 \setminus f[I]$ is connected. We enclose $f[I]$ in a rectangle $R \supset f[I]$ (Exercise 12) such that

$$\partial R \cap f[I] = \{\overline{p_1},\, \overline{p_2}\}$$

where ∂R is the (rectangular) boundary of R and $\overline{p_1}$ (resp. $\overline{p_2}$) is an interior point of the bottom (resp. top) side of R. Let U be the part of ∂R between $\overline{p_1}$ and $\overline{p_2}$ containing the right side of R. We may assume that

$$a \le t = f^{-1}(\overline{p_1}) < t' = f^{-1}(\overline{p_2}) \le b$$

where at least one $\le$ is strict. We split f in two halves, the arcs

$$f_1 = f \mid I' = [t,\, t'] \ \text{ and } \ f_2 = f_{\mathrm{e}} \mid I'' = [t',\, t + b - a]\ .$$

Let $S \subset R$ be any segment parallel to the bottom side of R and with endpoints in the interiors of the left and right sides of R. It follows (Exercise 13) that there exists a subsegment $T \subset S$ with endpoints $\overline{q_2} \in f_1[I']$ and $\overline{q_3} \in f_2[I'']$ and with interior disjoint to $f[I]$. From the assumption that $\mathbb{R}^2 \setminus f[I]$ is connected and from Theorem 6 it follows (Exercise 14) that there is a PL arc $f_{3,1}$ with image disjoint to $f[I]$, interior disjoint to $T \cup U$ and joining a point $\overline{p_3}$ in the interior of T to a point $\overline{q_1}$ in the interior of U.

We describe the nine arcs forming a $K_{3,3}$-conf. The arc $f_{1,1}$ is the part of U from $\overline{p_1}$ to $\overline{q_1}$, $f_{1,2}$ is the initial part of f_1 from $\overline{p_1}$ to $\overline{q_2}$, $f_{1,3}$ is the reversed final part of f_2 from $\overline{q_3}$ to $\overline{p_1}$, $f_{2,1}$ is the the part of U from $\overline{p_2}$ to $\overline{q_1}$, $f_{2,2}$ is the reversed final part of f_1 from $\overline{q_2}$ to $\overline{p_2}$, $f_{2,3}$ is the initial part of f_2 from $\overline{p_2}$ to $\overline{q_3}$, $f_{3,1}$ was already defined above, and $f_{3,2}$ (resp. $f_{3,3}$) is the straight segment joining $\overline{p_3}$ to $\overline{q_2}$ (resp. $\overline{q_3}$). It follows from these definitions that the required disjointness conditions hold and we indeed have a $K_{3,3}$-conf.

Exercise 12 *Explain how to find the rectangle R.*

Exercise 13 *Show that the subsegment T exists. Hint: intermediate values of continuous functions.*

Exercise 14 *Show that the arc $f_{3,1}$ exists.*

• *The second reduction $\exists$ $K_{3,3}$-conf $\Rightarrow$ $\exists$* PL *$K_{3,3}$-conf.* See `https://kam.mff.cuni.cz/~klazar/JordanPic3.pdf`
Suppose that $f_{i,j}$, $i,j \in [3]$, is a $K_{3,3}$-conf as obtained above, with the endpoints $\overline{p_i}$ and $\overline{q_j}$. Let $O_{i,j} \subset \mathbb{R}^2$ be the image of $f_{i,j}$ and $d > 0$ be the minimum of the distances between two of the six endpoints and between $O_{i,j}$ and an endpoint different from $\overline{p_i}$ and $\overline{q_j}$. We take the six closed discs

$$D(i) = \overline{B}(\overline{p_i},\, d/3) \ \text{ and } \ E(j) = \overline{B}(\overline{q_j},\, d/3),\ i,\, j \in [3]\ .$$

Any two discs have distance $\geq d/3$. It follows (Exercise 15) that for every $i,j \in [3]$ there exists the last time $t_{i,j} \in \mathbb{R}$ when $f_{i,j}$ exits $D(i)$ and the first following time $u_{i,j} > t_{i,j}$ when $f_{i,j}$ enters $E(j)$. It follows that for every $k,l \in [3]$,

$$f_{i,j}[(t_{i,j},\, u_{i,j})] \cap (D(k) \cup E(l)) = \emptyset\ .$$

The exit and entrance points (which lie on the boundaries of $D(i)$ and $E(j)$, respectively) are

$$\overline{l_{i,j}} = f_{i,j}(t_{i,j}) \in \partial D(i) \text{ and } \overline{e_{i,j}} = f_{i,j}(u_{i,j}) \in \partial E(j) \ .$$

For $i, j \in [3]$ we define the arc

$$f^{(1)}_{i,j}\colon J_{i,j} = [t'_{i,j},\, u'_{i,j}] \to \mathbb{R}^2, \text{ for some } t'_{i,j} < t_{i,j} \text{ and } u'_{i,j} > u_{i,j} \ ,$$

so that on $[t'_{i,j}, t_{i,j}]$ the arc $f^{(1)}_{i,j}$ is the segment $s(\overline{p_i}, \overline{l_{i,j}})$, on $I_{i,j} = [t_{i,j}, u_{i,j}]$ it coincides with $f_{i,j}$ and on $[u_{i,j}, u'_{i,j}]$ it is the segment $s(\overline{e_{i,j}}, \overline{q_j})$. We consider the minimum distance

$$\begin{aligned} e = \min\big(\{e_2(f_{i,j}[I_{i,j}],\, f^{(1)}_{k,l}[J_{k,l}]) \mid \\ \mid i,\, j,\, k,\, l \in [3],\, (i,\, j) \neq (k,\, l)\}\big) > 0 \end{aligned}$$

(Exercise 16). The restricted arcs $f_{i,j}\colon I_{i,j} \to \mathbb{R}^2$ are uniformly continuous (Exercise 17) and therefore $\exists\ \delta > 0$ such that for every $i, j \in [3]$ and every $t, u \in I_{i,j}$,

$$|t - u| \le \delta \Rightarrow e_2(f_{i,j}(t) - f_{i,j}(u)) \le \min(\{e/6,\, d/6\}) \ .$$

For any $i, j \in [3]$ we take a partition $t_{i,j} = v_0 < v_1 < \cdots < v_n = u_{i,j}$ of $I_{i,j}$ (its dependence on i, j is not marked) such that $v_k - v_{k-1} \le \delta$ and define

$$f^{(2)}_{i,j}\colon J_{i,j} \to \mathbb{R}^2$$

as $f^{(2)}_{i,j} = f^{(1)}_{i,j}$ on $[t'_{i,j}, t_{i,j}] \cup [u_{i,j}, u'_{i,j}]$ and as the PL map with the segments $s(f_{i,j}(v_{r-1}), f_{i,j}(v_r))$, $r = 1, 2, \ldots, n$, on $[t_{i,j},\, u_{i,j}]$. It follows from the choice of δ that the interiors $f^{(2)}_{i,j}[I_{i,j}]^0$ are pairwise disjoint, because for every $i, j, k, l \in [3]$ with $(i, j) \neq (k, l)$ one has that

$$e_2\big(f^{(2)}_{i,j}[I_{i,j}],\, f^{(2)}_{k,l}[J_{k,l}]\big) \ge e/3 \ ,$$

and that they are also disjoint to all six endpoints (Exercise 18). Finally, using Proposition 9 we replace the PL maps $f_{i,j}^{(2)}$ with the PL arcs

$$f_{i,j}^{(3)}\colon J_{i,j} \to \mathbb{R}^2$$

such that $f_{i,j}^{(3)}[J_{i,j}] \subset f_{i,j}^{(2)}[J_{i,j}]$ and that $f_{i,j}^{(3)}$ joins $\overline{p_i}$ to $\overline{q_j}$. It follows that the PL arcs $f_{i,j}^{(3)}$, $i, j \in [3]$, form a PL $K_{3,3}$ configuration.

Exercise 15 *Prove that the exit and entrance times for the arcs $f_{i,j}$ with respect to the discs $D(i)$ and $E(j)$ exist.*

Exercise 16 *Prove that the distance e is positive.*

Exercise 17 *Why are the restricted arcs $f_{i,j}\colon I_{i,j} \to \mathbb{R}^2$ uniformly continuous?*

Exercise 18 *Explain why are the interiors $f_{i,j}^{(2)}[I_{i,j}]^0$ pairwise disjoint and disjoint to the six endpoints.*

• *The third reduction* $\exists$ PL $K_{3,3}$*-conf* $\Rightarrow$ $\exists$ PL C*-conf.* See
`https://kam.mff.cuni.cz/~klazar/JordanPic4.pdf`
We show that any PL $K_{3,3}$-conf contains as a subgraph a PL C-conf.

Let f be an oriented PL circuit. Each segment s of f then determines the *right open halfplane* $\mathrm{rp}(s) \subset \mathbb{R}^2$ of points in $\mathbb{R}^2$ lying to the right of the line extending s. We similarly define the *left open halfplane* $\mathrm{lp}(s) \subset \mathbb{R}^2$. For $n \in \mathbb{N}$ the *right shadow* $r(s, n)$ of s is the segment $s' \subset \mathrm{rp}(s)$ whose endpoints are the two points in $\mathrm{rp}(s)$ that lie on the two axes of the two endpoints (corners) of s in distance $1/n$ from the endpoint of s. We define the *left shadow* $l(s, n)$, $n \in \mathbb{N}$, of s in the same way, only $\mathrm{rp}(s)$ is replaced with

$\mathrm{lp}(s)$. For $n \in \mathbb{N}$ we define the *right shadow* $r(f,n)$ and the *left shadow* $l(f,n)$ of the oriented PL circuit f by

$$r(f,\, n) = \bigcup_{s \in S(f)} r(s,\, n) \ \text{ and } \ l(f,\, n) = \bigcup_{s \in S(f)} l(s,\, n)$$

where $S(f)$ is the set of segments of f.

Proposition 19 (on shadows 1) *$\forall$ oriented* PL *circuit f and $\forall\, n$, both shadows $r(f,n)$ and $l(f,n)$ are images of* PL *maps.*

Proof. This is immediate from their definitions. □

Proposition 20 (on shadows 2) *Let $f\colon I \to \mathbb{R}^2$ be an oriented* PL *circuit. There is an n_0 such that for every $n \geq n_0$,*

$$r(f,\, n) \cap f[I] = \emptyset = l(f,\, n) \cap f[I]\ .$$

Proof. Let f be as stated and $d = \min_{s,s'} e_2(s, s') > 0$ where s, s' run through all pairs of segments of f with $s \cap s' = \emptyset$. Let s be any segment of f. It suffices to prove that for n large enough, $r(s,n) \cap f[I] = \emptyset$; for $l(s,n)$ the arguments is similar. Let s' and s'' be the two segments of f adjacent to s. It is easy to see that $r(s,n) \cap s = \emptyset$ for every n and that $r(s,n) \cap (s' \cup s'') = \emptyset$ for every large n (Exercise 21). Also,

$$r(s,n) \subset \{\overline{p} \in \mathbb{R}^2 \mid e_2(\{\overline{p}\}, s) \leq 1/n\}\ .$$

Thus it suffices to take n so large that $r(s,n) \cap (s' \cup s'') = \emptyset$ and that $1/n \leq d/3$. □

Exercise 21 *Show that for $n \geq n_0$, neither $r(s,n)$ nor $l(s,n)$ intersects the two segments of the* PL *circuit adjacent to s.*

Proposition 22 (on shadows 3) *Let $f\colon I \to \mathbb{R}^2$ be an oriented* PL *circuit. Then for any point $\overline{p} \in \mathbb{R}^2 \setminus f[I]$ one of two cases occurs.*

(L) For every $n \geq n_0$ a PL *arc in $\mathbb{R}^2 \setminus f[I]$ joins the point $\overline{p}$ to a point in $l(f, n)$.*

(R) For every $n \geq n_0$ a PL *arc in $\mathbb{R}^2 \setminus f[I]$ joins the point $\overline{p}$ to a point in $r(f, n)$.*

Proof. Let f and $\overline{p}$ be as stated and let u be any segment realizing the distance between $\overline{p}$ and $f[I]$. Then $\overline{p}$ is one endpoint of u, the other one $\overline{q} \in f[I]$ and $u^0 \subset \mathbb{R}^2 \setminus f[I]$. Considering u near $\overline{q}$, we see that (L) or (R) occurs. □

For an oriented PL circuit f we define $A_{f,R} \subset \mathbb{R}^2$ (resp. $A_{f,L} \subset \mathbb{R}^2$) as those points $\overline{p}$ in the complement of the image of f for which the above case (R) (resp. (L)) holds.

Corollary 23 (left and right sides) *Let $f\colon I \to \mathbb{R}^2$ be an oriented* PL *circuit. Then*

$$\mathbb{R}^2 \setminus f[I] = A_{f,R} \cup A_{f,L}$$

and $A_{f,R}$ and $A_{f,L}$ are connected open sets.

Proof. It is clear that $A_{f,R}$ and $A_{f,L}$ are open sets. Indeed, let $\overline{p} \in A_{f,R}$, say, witnessed by a PL arc g joining $\overline{p}$ in the complement of $f[I]$ to a point in $r(f, n)$. Let the ball $B = B(\overline{p}, r)$ have radius $r > 0$ so small that $B \subset \mathbb{R}^2 \setminus f[I]$ and that B intersect only one segment of g. Then for every $\overline{q} \in B$ we can easily modify g to a PL arc joining $\overline{q}$ in the complement of $f[I]$ to the same point in $r(f, n)$.

Let $\overline{p}, \overline{q} \in A_{f,R}$ be two distinct points (for $A_{f,L}$ the argument is similar). We show that there is a PL arc g that joins $\overline{p}$ to $\overline{q}$ in $\mathbb{R}^2 \setminus f[I]$. Then $A_{f,R}$ is connected by Theorem 7. We use Propositions 20 and 22 and the definition of the set $A_{f,R}$ and take large enough n such that $r(f,n) \cap f[I] = \emptyset$ and that there are PL arcs joining, respectively, $\overline{p}$ and $\overline{q}$ in $\mathbb{R}^2 \setminus f[I]$ to points in $r(f,n)$. By Propositions 9 and 19 there exists the required PL arc g. $\square$

Now suppose that the nine arcs $f_{i,j}$, $i,j \in [3]$, form a PL $K_{3,3}$-conf. Let $k\colon I \to \mathbb{R}^2$ be the oriented PL circuit formed by the six arcs $f_{1,1}$, $f_{2,1}$, $f_{2,2}$, $f_{3,2}$, $f_{3,3}$ and $f_{1,3}$. We denote the remaining three arcs by $e = f_{1,2}$, $g = f_{2,3}$ and $h = f_{3,1}$. We write $\mathbb{R}^2 \setminus k[I] = A_{k,R} \cup A_{k,L}$ as in Corollary 23. If $A_{k,R}$ and $A_{k,L}$ intersect then k is a PL C-conf and we are done. Hence these sets are disjoint. Then the interior of each of the arcs e, g and h lies completely in $A_{k,R}$ or completely in $A_{k,L}$ (else $A_{k,R}$ and $A_{k,L}$ would cut the interior of the arc, which is however a connected set). Thus two of these interiors lie in the same set, for example (other cases are similar) the interiors of e and h lie in $A_{k,R}$. We consider the oriented PL circuit $l\colon I' \to \mathbb{R}^2$ formed by the arcs $f_{2,2}$, $f_{2,1}$, $f_{1,1}$ and e; we orient the segments in e consistently with those in the other three PL arcs. But we see that the interior h^0 of h intersects both $A_{l,R}$ and $A_{l,L}$ (Exercise 24). By Corollary 23,

$$\mathbb{R}^2 \setminus l[I'] = A_{l,R} \cup h^0 \cup A_{l,L}$$

is connected and l is a PL C-conf (Exercise 25).

Exercise 24 *Why does h^0 intersect both the right and the left side of the oriented* PL *circuit l?*

Exercise 25 *Why is the set $A_{l,R} \cup h^0 \cup A_{l,L}$ connected?*

• *The fourth reduction* $\exists$ PL *C-conf* $\Rightarrow 0 = 1$. See the picture
`https://kam.mff.cuni.cz/~klazar/JordanPic5.pdf`
We suppose that $f\colon I \to \mathbb{R}^2$ is a PL circuit with connected complement $\mathbb{R}^2 \setminus f[I]$ and deduce a contradiction. It easily follows from the next proposition.

Proposition 26 (in or out?) *Let $f\colon I \to \mathbb{R}^2$ be a* PL *circuit and $D = \mathbb{R}^2 \setminus f[I]$. There exists a continuous map*

$$g\colon D \to \{0,\,1\} \subset \mathbb{R}$$

such that $g[D] = \{0, 1\}$.

Proof. Let f and D be as stated. We may assume (Exercise 28) that none of the finitely many segments of $f[I]$ is vertical. For any point $\overline{p} = (p_x, p_y) \in D$ we denote by

$$r(\overline{p}) = (x = p_x,\ y \geq p_y) \subset \mathbb{R}^2$$

the vertical ray (half-line) emanating upwards from $\overline{p}$. We define the function $g\colon D \to \{0,1\}$ as the parity of the finite sum (Exercise 29)

$$g(\overline{p}) = \sum_{\overline{q} \in r(\overline{p}) \cap f[I]} m(\overline{q}) \bmod 2$$

(for empty intersection the sum is 0), where the multiplicity $m(\overline{q}) \in \{0,1\}$ of the intersection $\overline{q}$ of the ray $r(\overline{p})$ and the graph $f[I]$ of the circuit f is defined as follows. If $\overline{q}$ is transversal, meaning that $f[I]$ lies locally near $\overline{q}$ on both sides of $r(\overline{p})$, we set $m(\overline{q}) = 1$. Else, when $f[I]$ lies locally near $\overline{q}$ only on one side of $r(\overline{p})$, we set $m(\overline{q}) = 0$. It follows that $m(\overline{q}) = 0$ iff $\overline{q}$ is the common corner of two consecutive segments of $f[I]$ lying on the same side of $r(\overline{p})$ (Exercise 30).

We prove that g is continuous. Let $\overline{p} = (p_x, p_y) \in D$ be an arbitrary point and $X \subset \mathbb{R}$ be the finite set of the x-coordinates of all corners of f, minus the number $\{p_x\}$. Let

$$\delta = \min\big(\{|p_x - a| \mid a \in X\} \cup \{e_2(\{\overline{p}\},\ f[I])\}\big) > 0\ .$$

We claim that

$$\overline{q} \in B(\overline{p},\ \delta) \Rightarrow g(\overline{q}) = g(\overline{p})\ .$$

To see it, compare for such $\overline{q}$ the finite intersections $X_1 = r(\overline{p}) \cap f[I]$ and $X_2 = r(\overline{q}) \cap f[I]$. By the choice of δ, when we move the point $\overline{p}$ to the position $\overline{q}$ every transversal intersection $\overline{r} \in X_1$ transforms in a transversal intersection $\overline{r'} \in X_2$, and every non-transversal intersection $\overline{r} \in X_1$ either disappears or remains the same non-transversal intersection $\overline{r} \in X_2$ or transforms in two distinct transversal intersections $\{\overline{r'}, \overline{r''}\} \subset X_2$. Also, all intersections in X_2 arise in these ways, no new intersection can appear. Thus $g(\overline{q}) = g(\overline{p})$ by the definition of g.

It remains to show that $g[D] = \{0, 1\}$. Let $\overline{q} \in f[I]$ be one of the highest corners of $f[I]$, i.e., with the maximum y-coordinate, and let ℓ be the axis of $\overline{q}$. Then it is easy to see that $g(\overline{p}) = 0$ for every point $\overline{p} \in \ell$ lying above $\overline{q}$ and that $g(\overline{p}) = 1$ for every point $\overline{p} \in \ell$ lying below $\overline{q}$ and sufficiently close to $\overline{q}$. $\square$

To obtain a contradiction for a PL circuit f possessing connected complement $D = \mathbb{R}^2 \setminus f[I]$, we take the function $g\colon D \to \{0, 1\}$ guaranteed by the previous proposition and take two points $\overline{p}$ and $\overline{q}$ in D such that $g(\overline{p}) = 0$ and $g(\overline{q}) = 1$. By Theorem 6 there is a PL arc $h\colon J \to \mathbb{R}^2$ with $h[J] \subset D$ and joining $\overline{p}$ to $\overline{q}$. But then $g(h)\colon J \to \{0, 1\}$ is a continuous function that maps the connected interval J to the disconnected set $g(h)[J] = \{0, 1\} \subset \mathbb{R}$, which is indeed a contradiction.

This completes the proof of Theorem 4.

Exercise 27 *Where do we run in difficulties when we attempt to define the function g in the same way for a* PL *arc? By Theorem 3, it has to fail somewhere.*

Exercise 28 *Why can we assume that none of the segments of the* PL *circuit f is vertical?*

Exercise 29 *Why is the displayed sum in the proof finite?*

Exercise 30 *Prove the equivalence characterizing geometricly the intersection points $\overline{q}$ with multiplicity zero.*

• *Concluding and other remarks.* The full Jordan theorem, which Jordan basically correctly proved (as discussed by Hales in `http://mizar.org/trybulec65/4.pdf`) in the book *Course d'analyse de l'École Polytechnique*, Paris, 1893, is as follows.

Theorem 31 (the Full Jordan Theorem) *For every plane circuit $f\colon I \to \mathbb{R}^2$ it holds that*

$$\mathbb{R}^2 \setminus f[I] = A_{\mathrm{int}} \cup A_{\mathrm{ext}}$$

where A_{int} and A_{ext} are nonempty open connected sets that are disjoint. Moreover, A_{int} is bounded and A_{ext} is unbounded.

Exercise 32 *How does the fact that $\mathbb{R}^2 \setminus f[I]$ is disconnected (i.e., Theorem 4 we have just proven) follow from the theorem?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 5, 13, 16 and 27.