MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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# **LECTURE 5 (March 15, 2023)** A PROOF OF WEAK JORDAN'S (CIRCUIT) THEOREM A LÀ THOMASSEN

I show you a proof modeled after the remarkable proof *Carsten Thomassen (1948)* gave in 1992 for the classical and famous theorem of *Camille Jordan (1838–1922)* in plane topology. Namely, I shall prove that any topological circuit disconnects the plane $\mathbb{R}^2$. Instead of Thomassen's original article I follow, to some extent, the article The Jordan Curve Theorem, Formally and Informally, *Amer. Math. Monthly* **114** (2007), 882–894 by *Thomas C. Hales (1958)*.

**Exercise 1** *Learn about achievements of these three excellent mathematicians.*

• *Arcs, circuits,* PL *maps and two theorems.* Let $I = I_{a,b} := [a, b]$ where $a < b$ are real numbers. An *arc* is an injective continuous map

$$f\colon I \to \mathbb{R}^2 \ .$$

Here $I$ is a subspace of the MS $(\mathbb{R}, e_1)$ and $\mathbb{R}^2$ is the MS $(\mathbb{R}^2, e_2)$. We call the points $f(a)$ and $f(b)$ in $\mathbb{R}^2$ the *endpoints of* $f$ and say that $f$ *joins* $f(a)$ *to* $f(b)$. If $f[I] \subset X \subset \mathbb{R}^2$, we say that $f$ *joins* $f(a)$ *to* $f(b)$ *in* $X$. The set $f^0 = f[I]^0 := f[(a, b)]$ is the *interior (of the arc* $f$*)*.

We define the *distance between two sets* $A, B \subset \mathbb{R}^2$ as the infimum

$$e_2(A,\, B) := \inf(\{e_2(\overline{x},\, \overline{y}) \mid \overline{x} \in A,\, \overline{y} \in B\}) \ .$$

If $A$ and $B$ are disjoint and compact, their distance is positive.

A *circuit* is any continuous map $f\colon I = I_{a,b} \to \mathbb{R}^2$

$$\text{such that } \forall\, x < y \text{ in } I\ \big(f(x) = f(y) \iff x = a \text{ and } y = b\big)\ .$$

We extend $f$ to the $(b-a)$-periodic function $f_{\mathrm{e}}\colon \mathbb{R} \to \mathbb{R}^2$ which coincides with $f$ on $[a, b]$.

A map $f\colon I \to \mathbb{R}^2$ is PL (*piece-wise linear*) if there exist a partition $a = t_0 < t_1 < \dots < t_n = b$, $n \in \mathbb{N}$, of $I$ and $n$ constants $\overline{s_i}, \overline{c_i} \in \mathbb{R}^2$, $i = 1, 2, \dots, n$ and no $\overline{s_i} = (0, 0)$, such that

$$\forall\, i \in [n]\ \forall\, t \in [t_{i-1},\, t_i]\ \big(f(t) = t \cdot \overline{s_i} + \overline{c_i}\big)\ .$$

The points $f(t_i) \in \mathbb{R}^2$, $i = 0, 1, \dots, n$, are the *corners (of $f$)*. The plane straight segments $f[\,[t_{i-1}, t_i]\,]$ are called *segments (of $f$)*. For any two points $\overline{p}, \overline{q} \in \mathbb{R}^2$ we denote by $s(\overline{p}, \overline{q})$ the straight segment joining them. An *axis* of a corner $f(t_i)$ with $0 < i < n$ is the line going through $f(t_i)$ that halves both angles at $f(t_i)$ (determined by the two segments of $f$ incident with the corner). If $f(t_0) = f(t_n)$, this corner also has an axis. An *oriented* PL *circuit* is one where all segments are oriented consistently in one of the two ways.

**Exercise 2** *Show that any* PL *map is continuous.*

Two famous theorems about images of arcs and circuits are

**Theorem 3 (Arc T.)** *For any arc $f\colon I \to \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is connected.*

and (`https://kam.mff.cuni.cz/~klazar/JordanPic1.pdf`)

**Theorem 4 (Weak Jordan's T.)** *For any circuit $f\colon I \to \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is disconnected.*

Our goal for this lecture is to prove the latter theorem. We discuss the full Jordan's Theorem in the concluding remarks. Not very convincing outline of a proof of the Arc Theorem is given in the article of Hales on pp. 890–891.

**Exercise 5** *Prove that the complements in the two theorems are open sets in the plane.*

• *Two results on connected sets.* Recall that $X \subset \mathbb{R}^2$ is connected if there do not exist two open (or closed) sets $A, B \subset \mathbb{R}^2$ cutting $X$, i.e., such that $X \subset A \cup B$, both intersections $X \cap A$ and $X \cap B$ are nonempty and $X \cap A \cap B = \emptyset$. We say that $X \subset \mathbb{R}^2$ is PL-*connected* if for every two distinct points $x, y \in X$ there is a PL arc $f\colon I \to \mathbb{R}^2$ such that $f$ joins $x$ to $y$ in $X$.

**Theorem 6 (conn. $\leftrightarrow$ PL-conn.)** *Any open set $X \subset \mathbb{R}^2$ is connected if and only if it is* PL-*connected.*

**Proof.** $\neg \Rightarrow \neg$. Suppose that $A$ and $B$ cut $X$, $x \in X \cap A$, $y \in X \cap B$ and that the PL arc $f\colon I \to \mathbb{R}^2$ with $f[I] \subset X$ joins $x$ to $y$. Then $A$ and $B$ cut $f[I]$ and $f[I]$ is disconnected. This is impossible because $f[I]$ is connected as a continuous image of the connected interval $I$. Hence $f$ does not exist.

$\neg \Leftarrow \neg$. Consider the partition $X/\sim$ of $X$ by the equivalence relation $\sim$ on $X$ (Exercise 8) defined by $x \sim y$ iff a PL arc joins $x$ to $y$ in $X$. It is not hard to see that every block $A \in X/\sim$ is open. We assume that $X$ is not PL-connected: $|X/\sim| \geq 2$. We take any block $A \in X/\sim$ and define

$$B := \bigcup \left((X/\sim) \setminus \{A\}\right) .$$

Then $A$ and $B$ are open sets cutting $X$ and $X$ is disconnected. $\square$

**Exercise 7** *Describe a set $X \subset \mathbb{R}^2$ such that* (i) *$X$ is a countable union of plane segments,* (ii) *$X$ is connected and* (iii) *$X$ is not* PL*-connected.*

**Exercise 8** *Show that the relation $\sim$ in the previous proof is transitive. Or, better, prove the next more general proposition.*

**Proposition 9 (PL maps and PL arcs)** *For every* PL *map $f\colon I = I_{a,b} \to \mathbb{R}^2$ there is a* PL *arc $g\colon I' \to \mathbb{R}^2$ such that $g[I'] \subset f[I]$ and $g$ joins $f(a)$ to $f(b)$.*

• (PL) *configurations.* A *$C$-configuration*, abbreviated *$C$-conf*, is any circuit $f\colon I \to \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus f[I]$ is connected. Our goal is to prove that no $C$-conf exists, i.e., that Theorem 4 holds. A *$K_{3,3}$-configuration*, abbreviated *$K_{3,3}$-conf*, is any nine-tuple of arcs $f_{i,j}$, $i, j \in [3]$, such that their endpoints form a six-element set

$$K := \{\overline{p_1}, \overline{p_2}, \overline{p_3}, \overline{q_1}, \overline{q_2}, \overline{q_3}\} \subset \mathbb{R}^2 ,$$

for every pair $i, j \in [3]$ the arc $f_{i,j}$ joins $\overline{p_i}$ to $\overline{q_j}$, and the nine interiors $f^0_{i,j}$ are pairwise disjoint and disjoint to $K$. Graph-theoretically, a $K_{3,3}$-conf is a plane drawing (i.e., without crossings) of the complete bipartite graph $K_{3,3}$.

**Exercise 10** *Explain why no $K_{3,3}$-conf exists. Hint: recall the course* Discrete Mathematics.

A PL *$C$-configuration*, abbreviated PL *$C$-conf*, is any $C$-conf in which the circuit $f$ is a PL map. Similarly, a PL *$K_{3,3}$-configuration*, abbreviated PL *$K_{3,3}$-conf*, is any $K_{3,3}$-conf in which all nine arcs $f_{i,j}$ are PL maps. We begin our proof of Theorem 4.

• *Thomassen's reduction(s).* We divide it in four reductions (implications).

1. $\exists$ $C$-conf $\Rightarrow \exists$ $K_{3,3}$-conf
2. $\exists$ $K_{3,3}$-conf $\Rightarrow \exists$ PL $K_{3,3}$-conf
3. $\exists$ PL $K_{3,3}$-conf $\Rightarrow \exists$ PL $C$-conf
4. $\exists$ PL $C$-conf $\Rightarrow 0 = 1$

When we prove these four implications, Theorem 4 will follow. The main invention of Thomassen is the first reduction.

**Exercise 11** *Exercise 10 says that $\exists$ $K_{3,3}$-conf $\Rightarrow 0 = 1$. Does not this simplify our proof?*

• *The first reduction $\exists$ $C$-conf $\Rightarrow \exists$ $K_{3,3}$-conf.* See the picture `https://kam.mff.cuni.cz/~klazar/JordanPic2.pdf`
Let $f\colon I = I_{a,b} \to \mathbb{R}^2$ be a $C$-conf, i.e., $f$ is a circuit such that the open set $\mathbb{R}^2 \setminus f[I]$ is connected. We enclose $f[I]$ in a rectangle $R \supset f[I]$ (Exercise 12) such that

$$\partial R \cap f[I] = \{\overline{p_1}, \overline{p_2}\}$$

where $\partial R$ is the (rectangular) boundary of $R$ and $\overline{p_1}$ (resp. $\overline{p_2}$) is an interior point of the bottom (resp. top) side of $R$. Let $U$ be the part of $\partial R$ between $\overline{p_1}$ and $\overline{p_2}$ containing the right side of $R$. We may assume that

$$a \le t := f^{-1}(\overline{p_1}) < t' := f^{-1}(\overline{p_2}) \le b$$

where at least one $\le$ is strict. We split $f$ in two halves, the arcs

$$f_1 := f \,|\, I' := [t,\, t'] \;\text{ and }\; f_2 := f_{\mathrm{e}} \,|\, I'' := [t',\, t + b - a]\;.$$

Let $S \subset R$ be any segment parallel to the bottom side of $R$ and with endpoints in the interiors of the left and right sides of $R$. It follows (Exercise 13) that there exists a subsegment $T \subset S$ with endpoints $\overline{q_2} \in f_1[I']$ and $\overline{q_3} \in f_2[I'']$ and with interior disjoint to $f[I]$. From the assumption that $\mathbb{R}^2 \setminus f[I]$ is connected and from Theorem 6 it follows (Exercise 14) that there is a PL arc $f_{3,1}$ with image disjoint to $f[I]$, interior disjoint to $T \cup U$ and joining a point $\overline{p_3}$ in the interior of $T$ to a point $\overline{q_1}$ in the interior of $U$.

We describe the nine arcs forming a $K_{3,3}$-conf. The arc $f_{1,1}$ is the part of $U$ from $\overline{p_1}$ to $\overline{q_1}$, $f_{1,2}$ is the initial part of $f_1$ from $\overline{p_1}$ to $\overline{q_2}$, $f_{1,3}$ is the reversed final part of $f_2$ from $\overline{q_3}$ to $\overline{p_1}$, $f_{2,1}$ is the the part of $U$ from $\overline{p_2}$ to $\overline{q_1}$, $f_{2,2}$ is the reversed final part of $f_1$ from $\overline{q_2}$ to $\overline{p_2}$, $f_{2,3}$ is the initial part of $f_2$ from $\overline{p_2}$ to $\overline{q_3}$, $f_{3,1}$ was already defined above, and $f_{3,2}$ (resp. $f_{3,3}$) is the straight segment joining $\overline{p_3}$ to $\overline{q_2}$ (resp. $\overline{q_3}$). It follows from these definitions that the required disjointness conditions hold and we indeed have a $K_{3,3}$-conf.

**Exercise 12** *Explain how to find the rectangle $R$.*

**Exercise 13** *Show that the subsegment $T$ exists. Hint: intermediate values of continuous functions.*

**Exercise 14** *Show that the arc $f_{3,1}$ exists.*

• *The second reduction* $\exists$ $K_{3,3}$*-conf* $\Rightarrow$ $\exists$ PL $K_{3,3}$*-conf.* See
`https://kam.mff.cuni.cz/~klazar/JordanPic3.pdf`
Suppose that $f_{i,j}$, $i, j \in [3]$, is a $K_{3,3}$-conf as obtained above, with the endpoints $\overline{p_i}$ and $\overline{q_j}$. Let $O_{i,j} \subset \mathbb{R}^2$ be the image of $f_{i,j}$ and $d > 0$ be the minimum of the distances between two of the six endpoints and between $O_{i,j}$ and an endpoint different from $\overline{p_i}$ and $\overline{q_j}$. We take the six closed discs

$$D(i) := \overline{B}(\overline{p_i},\, d/3) \ \text{ and } \ E(j) := \overline{B}(\overline{q_j},\, d/3),\ i,\, j \in [3]\ .$$

Any two discs have distance $\geq d/3$. It follows (Exercise 15) that for every $i, j \in [3]$ there exists the last time $t_{i,j} \in \mathbb{R}$ when $f_{i,j}$ exits $D(i)$ and the first following time $u_{i,j} > t_{i,j}$ when $f_{i,j}$ enters $E(j)$. It follows that for every $k, l \in [3]$,

$$f_{i,j}[(t_{i,j},\, u_{i,j})] \cap (D(k) \cup E(l)) = \emptyset \;.$$

The exit and entrance points (which lie on the boundaries of $D(i)$ and $E(j)$, respectively) are

$$\overline{l_{i,j}} := f_{i,j}(t_{i,j}) \in \partial D(i) \;\text{ and }\; \overline{e_{i,j}} := f_{i,j}(u_{i,j}) \in \partial E(j) \;.$$

For $i, j \in [3]$ we define the arc

$$f^{(1)}_{i,j}\colon J_{i,j} := [t'_{i,j},\, u'_{i,j}] \to \mathbb{R}^2, \text{ for some } t'_{i,j} < t_{i,j} \text{ and } u'_{i,j} > u_{i,j} \;,$$

so that on $[t'_{i,j}, t_{i,j}]$ the arc $f^{(1)}_{i,j}$ is the segment $s(\overline{p_i}, \overline{l_{i,j}})$, on $I_{i,j} := [t_{i,j}, u_{i,j}]$ it coincides with $f_{i,j}$ and on $[u_{i,j}, u'_{i,j}]$ it is the segment $s(\overline{e_{i,j}}, \overline{q_j})$. We consider the minimum distance

$$\begin{aligned} e := \min\big(\{e_2(f_{i,j}[I_{i,j}],\, f^{(1)}_{k,l}[J_{k,l}]) \mid \\ \mid i,\, j,\, k,\, l \in [3],\, (i,\, j) \neq (k,\, l)\}\big) > 0 \end{aligned}$$

(Exercise 16). The restricted arcs $f_{i,j}\colon I_{i,j} \to \mathbb{R}^2$ are uniformly continuous (Exercise 17) and therefore $\exists\, \delta > 0$ such that for every $i, j \in [3]$ and every $t, u \in I_{i,j}$,

$$|t-u| \leq \delta \Rightarrow e_2(f_{i,j}(t) - f_{i,j}(u)) \leq \min(\{e/6,\, d/6\}) \;.$$

For any $i, j \in [3]$ we take a partition $t_{i,j} = v_0 < v_1 < \cdots < v_n = u_{i,j}$ of $I_{i,j}$ (its dependence on $i, j$ is not marked) such that $v_k - v_{k-1} \leq \delta$ and define

$$f^{(2)}_{i,j}\colon J_{i,j} \to \mathbb{R}^2$$

as $f^{(2)}_{i,j} = f^{(1)}_{i,j}$ on $[t'_{i,j}, t_{i,j}] \cup [u_{i,j}, u'_{i,j}]$ and as the PL map with the segments $s(f_{i,j}(v_{r-1}), f_{i,j}(v_r))$, $r = 1, 2, \dots, n$, on $[t_{i,j},\, u_{i,j}]$. It follows from the choice of $\delta$ that the interiors $f^{(2)}_{i,j}[I_{i,j}]^0$ are pairwise disjoint, because for every $i, j, k, l \in [3]$ with $(i, j) \neq (k, l)$ one has that

$$e_2\big(f^{(2)}_{i,j}[I_{i,j}],\, f^{(2)}_{k,l}[J_{k,l}]\big) \geq e/3\ ,$$

and that they are also disjoint to all six endpoints (Exercise 18). Finally, using Proposition 9 we replace the PL maps $f^{(2)}_{i,j}$ with the PL arcs

$$f^{(3)}_{i,j}\colon\, J_{i,j} \to \mathbb{R}^2$$

such that $f^{(3)}_{i,j}[J_{i,j}] \subset f^{(2)}_{i,j}[J_{i,j}]$ and that $f^{(3)}_{i,j}$ joins $\overline{p_i}$ to $\overline{q_j}$. It follows that the PL arcs $f^{(3)}_{i,j}$, $i, j \in [3]$, form a PL $K_{3,3}$ configuration.

**Exercise 15** *Prove that the exit and entrance times for the arcs $f_{i,j}$ with respect to the discs $D(i)$ and $E(j)$ exist.*

**Exercise 16** *Prove that the distance $e$ is positive.*

**Exercise 17** *Why are the restricted arcs $f_{i,j}\colon I_{i,j} \to \mathbb{R}^2$ uniformly continuous?*

**Exercise 18** *Explain why are the interiors $f^{(2)}_{i,j}[I_{i,j}]^0$ pairwise disjoint and disjoint to the six endpoints.*

• *The third reduction* $\exists$ PL $K_{3,3}$*-conf* $\Rightarrow \exists$ PL $C$*-conf.* See
`https://kam.mff.cuni.cz/~klazar/JordanPic4.pdf`
We show that any PL $K_{3,3}$-conf contains as a subgraph a PL $C$-conf.

Let $f$ be an oriented PL circuit. Each segment $s$ of $f$ then determines the *right open halfplane* $\mathrm{rp}(s) \subset \mathbb{R}^2$ of points in $\mathbb{R}^2$ lying to the right of the line extending $s$. We similarly define the *left open halfplane* $\mathrm{lp}(s) \subset \mathbb{R}^2$. For $n \in \mathbb{N}$ the *right shadow* $r(s, n)$ of $s$ is the segment $s' \subset \mathrm{rp}(s)$ whose endpoints are the two points in $\mathrm{rp}(s)$ that lie on the two axes of the two endpoints (corners) of $s$ in distance $1/n$ from the endpoint of $s$. We define the *left shadow* $l(s, n)$, $n \in \mathbb{N}$, of $s$ in the same way, only $\mathrm{rp}(s)$ is replaced with $\mathrm{lp}(s)$. For $n \in \mathbb{N}$ we define the *right shadow* $r(f, n)$ and the *left shadow* $l(f, n)$ of the oriented PL circuit $f$ by

$$r(f,\, n) := \bigcup_{s \in S(f)} r(s,\, n) \ \text{ and } \ l(f,\, n) := \bigcup_{s \in S(f)} l(s,\, n)$$

where $S(f)$ is the set of segments of $f$.

**Proposition 19 (on shadows 1)** $\forall$ *oriented* PL *circuit* $f$ *and* $\forall\, n$, *both shadows* $r(f, n)$ *and* $l(f, n)$ *are images of* PL *maps.*

**Proof.** This is immediate from their definitions. □

**Proposition 20 (on shadows 2)** *Let* $f\colon I \to \mathbb{R}^2$ *be an oriented* PL *circuit. There is an* $n_0$ *such that for every* $n \geq n_0$,

$$r(f,\, n) \cap f[I] = \emptyset = l(f,\, n) \cap f[I]\ .$$

**Proof.** Let $f$ be as stated and $d := \min_{s,s'} e_2(s, s') > 0$ where $s, s'$ run through all pairs of segments of $f$ with $s \cap s' = \emptyset$. Let $s$ be any segment of $f$. It suffices to prove that for $n$ large enough, $r(s, n) \cap f[I] = \emptyset$; for $l(s, n)$ the arguments is similar. Let $s'$ and $s''$ be the two segments of $f$ adjacent to $s$. It is easy to see that $r(s, n) \cap s = \emptyset$ for every $n$ and that $r(s, n) \cap (s' \cup s'') = \emptyset$ for every large $n$ (Exercise 21). Also,

$$r(s, n) \subset \{\overline{p} \in \mathbb{R}^2 \mid e_2(\{\overline{p}\}, s) \leq 1/n\}\ .$$

Thus it suffices to take $n$ so large that $r(s,n) \cap (s' \cup s'') = \emptyset$ and that $1/n \leq d/3$. $\square$

**Exercise 21** *Show that for $n \geq n_0$, neither $r(s,n)$ nor $l(s,n)$ intersects the two segments of the* PL *circuit adjacent to $s$.*

**Proposition 22 (on shadows 3)** *Let $f\colon I \to \mathbb{R}^2$ be an oriented* PL *circuit. Then for any point $\overline{p} \in \mathbb{R}^2 \setminus f[I]$ one of two cases occurs.*

*(L) For every $n \geq n_0$ a* PL *arc in $\mathbb{R}^2 \setminus f[I]$ joins the point $\overline{p}$ to a point in $l(f,n)$.*

*(R) For every $n \geq n_0$ a* PL *arc in $\mathbb{R}^2 \setminus f[I]$ joins the point $\overline{p}$ to a point in $r(f,n)$.*

**Proof.** Let $f$ and $\overline{p}$ be as stated and let $u$ be any segment realizing the distance between $\overline{p}$ and $f[I]$. Then $\overline{p}$ is one endpoint of $u$, the other one $\overline{q} \in f[I]$ and $u^0 \subset \mathbb{R}^2 \setminus f[I]$. Considering $u$ near $\overline{q}$, we see that (L) or (R) occurs. $\square$

For an oriented PL circuit $f$ we define $A_{f,R} \subset \mathbb{R}^2$ (resp. $A_{f,L} \subset \mathbb{R}^2$) as those points $\overline{p}$ in the complement of the image of $f$ for which the above case (R) (resp. (L)) holds.

**Corollary 23 (left and right sides)** *Let $f\colon I \to \mathbb{R}^2$ be an oriented* PL *circuit. Then*

$$\mathbb{R}^2 \setminus f[I] = A_{f,R} \cup A_{f,L}$$

*and $A_{f,R}$ and $A_{f,L}$ are connected open sets.*

**Proof.** It is clear that $A_{f,R}$ and $A_{f,L}$ are open sets. Indeed, let $\overline{p} \in A_{f,R}$, say, witnessed by a PL arc $g$ joining $\overline{p}$ in the complement of $f[I]$ to a point in $r(f,n)$. Let the ball $B := B(\overline{p}, r)$ have radius $r > 0$ so small that $B \subset \mathbb{R}^2 \setminus f[I]$ and that $B$ intersect only one segment of $g$. Then for every $\overline{q} \in B$ we can easily modify $g$ to a PL arc joining $\overline{q}$ in the complement of $f[I]$ to the same point in $r(f,n)$.

Let $\overline{p}, \overline{q} \in A_{f,R}$ be two distinct points (for $A_{f,L}$ the argument is similar). We show that there is a PL arc $g$ that joins $\overline{p}$ to $\overline{q}$ in $\mathbb{R}^2 \setminus f[I]$. Then $A_{f,R}$ is connected by Theorem 7. We use Propositions 20 and 22 and the definition of the set $A_{f,R}$ and take large enough $n$ such that $r(f,n) \cap f[I] = \emptyset$ and that there are PL arcs joining, respectively, $\overline{p}$ and $\overline{q}$ in $\mathbb{R}^2 \setminus f[I]$ to points in $r(f,n)$. By Propositions 9 and 19 there exists the required PL arc $g$. $\square$

Now suppose that the nine arcs $f_{i,j}$, $i, j \in [3]$, form a PL $K_{3,3}$-conf. Let $k\colon I \to \mathbb{R}^2$ be the oriented PL circuit formed by the six arcs $f_{1,1}$, $f_{2,1}$, $f_{2,2}$, $f_{3,2}$, $f_{3,3}$ and $f_{1,3}$. We denote the remaining three arcs by $e := f_{1,2}$, $g := f_{2,3}$ and $h := f_{3,1}$. We write $\mathbb{R}^2 \setminus k[I] = A_{k,R} \cup A_{k,L}$ as in Corollary 23. If $A_{k,R}$ and $A_{k,L}$ intersect then $k$ is a PL $C$-conf and we are done. Hence these sets are disjoint. Then the interior of each of the arcs $e$, $g$ and $h$ lies completely in $A_{k,R}$ or completely in $A_{k,L}$ (else $A_{k,R}$ and $A_{k,L}$ would cut the interior of the arc, which is however a connected set). Thus two of these interiors lie in the same set, for example (other cases are similar) the interiors of $e$ and $h$ lie in $A_{k,R}$. We consider the oriented PL circuit $l\colon I' \to \mathbb{R}^2$ formed by the arcs $f_{1,3}$, $f_{3,3}$, $f_{3,2}$ and $e$; we orient the segments in $e$ consistently with those in the other three PL arcs. But we see that the interior $h^0$ of $h$ intersects both $A_{l,R}$ and $A_{l,L}$

(Exercise 24). By Corollary 23,

$$\mathbb{R}^2 \setminus l[I'] = A_{l,R} \cup h^0 \cup A_{l,L}$$

is connected and $l$ is a PL $C$-conf (Exercise 25).

**Exercise 24** *Why does $h^0$ intersect both the right and the left side of the oriented* PL *circuit $l$?*

**Exercise 25** *Why is the set $A_{l,R} \cup h^0 \cup A_{l,L}$ connected?*

• *The fourth reduction* $\exists$ PL *$C$-conf* $\Rightarrow 0 = 1$. See the picture
`https://kam.mff.cuni.cz/~klazar/JordanPic5.pdf`
We suppose that $f\colon I \to \mathbb{R}^2$ is a PL circuit with connected complement $\mathbb{R}^2 \setminus f[I]$ and deduce a contradiction. It easily follows from the next proposition.

**Proposition 26 (inside or outside?)** *Let $f\colon I \to \mathbb{R}^2$ be a* PL *circuit and $D := \mathbb{R}^2 \setminus f[I]$. There exists a continuous map*

$$g\colon D \to \{0,\, 1\} \subset \mathbb{R}$$

*such that $g[D] = \{0, 1\}$.*

**Proof.** Let $f$ and $D$ be as stated. We may assume (Exercise 28) that none of the finitely many segments of $f[I]$ is vertical. For any point $\overline{p} = (p_x, p_y) \in D$ we denote by

$$r(\overline{p}) = (x = p_x,\ y \ge p_y) \subset \mathbb{R}^2$$

the vertical ray (half-line) emanating upwards from $\overline{p}$. We define the function $g\colon D \to \{0, 1\}$ as the parity of the finite sum (Exercise 29)

$$g(\overline{p}) := \sum_{\overline{q} \in r(\overline{p}) \cap f[I]} m(\overline{q}) \bmod 2$$

(for empty intersection the sum is 0), where the multiplicity $m(\overline{q}) \in \{0, 1\}$ of the intersection $\overline{q}$ of the ray $r(\overline{p})$ and the graph $f[I]$ of the circuit $f$ is defined as follows. If $\overline{q}$ is transversal, meaning that $f[I]$ lies locally near $\overline{q}$ on both sides of $r(\overline{p})$, we set $m(\overline{q}) := 1$. Else, when $f[I]$ lies locally near $\overline{q}$ only on one side of $r(\overline{p})$, we set $m(\overline{q}) := 0$. It follows that $m(\overline{q}) = 0$ iff $\overline{q}$ is the common corner of two consecutive segments of $f[I]$ lying on the same side of $r(\overline{p})$ (Exercise 30).

We prove that $g$ is continuous. Let $\overline{p} = (p_x, p_y) \in D$ be an arbitrary point and $X \subset \mathbb{R}$ be the finite set of the $x$-coordinates of all corners of $f$, minus the number $\{p_x\}$. Let

$$\delta := \min\big(\{|p_x - a| \mid a \in X\} \cup \{e_2(\{\overline{p}\},\ f[I])\}\big) > 0\ .$$

We claim that

$$\overline{q} \in B(\overline{p},\ \delta) \Rightarrow g(\overline{q}) = g(\overline{p})\ .$$

To see it, compare for such $\overline{q}$ the finite intersections $X_1 := r(\overline{p}) \cap f[I]$ and $X_2 := r(\overline{q}) \cap f[I]$. By the choice of $\delta$, when we move the point $\overline{p}$ to the position $\overline{q}$ every transversal intersection $\overline{r} \in X_1$ transforms in a transversal intersection $\overline{r'} \in X_2$, and every non-transversal intersection $\overline{r} \in X_1$ either disappears or remains the same non-transversal intersection $\overline{r} \in X_2$ or transforms in two distinct transversal intersections $\{\overline{r'}, \overline{r''}\} \subset X_2$. Also, all intersections in $X_2$ arise in these ways, no new intersection can appear. Thus $g(\overline{q}) = g(\overline{p})$ by the definition of $g$.

It remains to show that $g[D] = \{0, 1\}$. Let $\overline{q} \in f[I]$ be one of the highest corners of $f[I]$, i.e., with the maximum $y$-coordinate, and let $\ell$ be the axis of $\overline{q}$. Then it is easy to see that $g(\overline{p}) = 0$ for every point $\overline{p} \in \ell$ lying above $\overline{q}$ and that $g(\overline{p}) = 1$ for every point $\overline{p} \in \ell$ lying below $\overline{q}$ and sufficiently close to $\overline{q}$. □

To obtain a contradiction for a PL circuit $f$ possessing connected complement $D = \mathbb{R}^2 \setminus f[I]$, we take the function $g\colon D \to \{0, 1\}$ guaranteed by the previous proposition and take two points $\overline{p}$ and $\overline{q}$ in $D$ such that $g(\overline{p}) = 0$ and $g(\overline{q}) = 1$. By Theorem 6 there is a PL arc $h\colon J \to \mathbb{R}^2$ with $h[J] \subset D$ and joining $\overline{p}$ to $\overline{q}$. But then $g(h)\colon J \to \{0, 1\}$ is a continuous function that maps the connected interval $J$ to the disconnected set $g(h)[J] = \{0, 1\} \subset \mathbb{R}$, which is indeed a contradiction.

This completes the proof of Theorem 4.

**Exercise 27** *Where do we run in difficulties when we attempt to define the function $g$ in the same way for a* PL *arc? By Theorem 3, it has to fail somewhere.*

**Exercise 28** *Why can we assume that none of the segments of the* PL *circuit $f$ is vertical?*

**Exercise 29** *Why is the displayed sum in the proof finite?*

**Exercise 30** *Prove the equivalence characterizing geometricly the intersection points $\overline{q}$ with multiplicity zero.*

• *Concluding and other remarks.* The full Jordan's theorem, which Jordan published and basically correctly proved (see `http://mizar.org/trybulec65/4.pdf` for detailed discussion by Hales of Jordan's proof) in his textbook *Course d'analyse de l'École Polytechnique*, Paris, 1893, is as follows.

**Theorem 31 (Full Jordan's T.)** *For any circuit $f\colon I \to \mathbb{R}^2$,*

$$\mathbb{R}^2 \setminus f[I] = A_{\mathrm{int}} \cup A_{\mathrm{ext}}$$

*where* $A_{\text{int}}$ *and* $A_{\text{ext}}$ *are nonempty open connected sets that are disjoint. Moreover,* $A_{\text{int}}$ *is bounded and* $A_{\text{ext}}$ *is unbounded.*

**Exercise 32** *How does the fact that* $\mathbb{R}^2 \setminus f[I]$ *is disconnected (i.e., Theorem 4 we have just proven) follow from the theorem?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 1, 5, 13, 16 and 27.