MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2023/24
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# LECTURE 2 (February 27, 2024) OSTROWSKI'S THEOREM. COMPACT METRIC SPACES.

• *Ostrowski's theorem.* On any field $F$ we have the *trivial norm.* It is a function $\|\cdot\|$ with $\|0_F\| = 0$ and $\|x\| = 1$ for $x \neq 0_F$.

**Exercise 1** *Prove that a trivial norm is a norm.*

From the usual absolute value $|\cdot|$ on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, we get many other norms by exponentiation.

**Exercise 2** *Prove that for any $c > 0$, $|\cdot|^c$ is a norm (on $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) if and only if $c \leq 1$. We will call this norm the modified absolute value.*

For $\alpha \in \mathbb{Q}$ and a prime $p$, the *canonical $p$-adic norm* $\|\cdot\|_p$ is defined by

$$\|\alpha\|_p = p^{-\mathrm{ord}_p(\alpha)}$$

− in the general $p$-adic norm $|\cdot|_p$ we set $c = 1/p$.

**Exercise 3** *Let $M = \{2, 3, 5, 7, 11, \dots\} \cup \{\infty\}$ and $\|\cdot\|_\infty = |\cdot|$ (ordinary absolute value). Prove that for every nonzero number* $\alpha \in \mathbb{Q}$ the product formula

$$\prod_{p \in M} \|\alpha\|_p = 1$$

*holds.*

**Exercise 4** *Let $\|\cdot\|$ be a nontrivial norm on the field $\mathbb{Q}$. Prove that $\exists\, n \in \mathbb{N} \big(n \geq 2 \wedge \|n\| \neq 1\big)$.*

**Exercise 5** *Prove that for every two coprime numbers $a, b \in \mathbb{Z}$ there exist numbers $c, d \in \mathbb{Z}$ such that*

$$ac + db = 1\ .$$

**Theorem 6 (A. Ostrowski, 1916)** *Let $\|\cdot\|$ be a norm on the field of rational numbers $\mathbb{Q}$. Then one of the following three cases occurs.*

1. *It is a trivial norm.*
2. *There exists a real $c \in (0, 1]$ such that $\|x\| = |x|^c$.*
3. *There exists a real $c \in (0, 1)$ and a prime number $p$ such that $\|x\| = |x|_p = c^{\operatorname{ord}_p(x)}$.*

*Modified absolute values and p-adic norms are therefore the only non-trivial norms on the field of rational numbers.*

**Proof.** Let $\|\cdot\|$ be a nontrivial norm. By Exercise 4 there exists an $n \in \mathbb{N} \setminus \{1\}$ such that $\|n\| \neq 1$. Two cases occur.

1. *There exists an $n \in \mathbb{N}$ such that $\|n\| > 1$.* Let $n_0$ be the smallest such $n$. Apparently $n_0 \geq 2$ a

$$1 \leq m < n_0 \Rightarrow \|m\| \leq 1\ . \tag{1}$$

There is a unique real number $c > 0$ such that

$$\|n_0\| = n_0^c\ . \tag{2}$$

Any $n \in \mathbb{N}$ expands in base $n_0$:

$$\begin{gathered} n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s \text{ where} \\ a_i,\ s \in \mathbb{N}_0,\ 0 \leq a_i < n_0 \ \text{ and } \ a_s \neq 0\ . \end{gathered}$$

For $n_0 = 10$ this is the usual decadic notation, like $2024 = 2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0$. So

$$\begin{aligned}
\|n\| &= \|a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s\| \\
&\overset{\Delta\text{-ineq. and multipl. of } \|\cdot\|}{\leq} \textstyle\sum_{j=0}^{s} \|a_j\| \cdot \|n_0\|^j \\
&\overset{\text{eq. (1) and (2)}}{\leq} \textstyle\sum_{j=0}^{s} n_0^{jc} \ \leq \ n_0^{sc} \sum_{i=0}^{\infty} (1/n_0^c)^i \\
&\overset{n_0^s \leq n}{\leq} n^c C \ \text{ where } \ C = \textstyle\sum_{i=0}^{\infty} (1/n_0^c)^i \ .
\end{aligned}$$

Hence

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \leq C n^c\right) . \tag{3}$$

This bound holds in fact even with $C = 1$. For each $m, n \in \mathbb{N}$, multiplicativity of the norm and inequality (3) give

$$\|n\|^m = \|n^m\| \leq C\,(n^m)^c = C\,(n^c)^m \ .$$

We take the $m$-th root and get that $\|n\| \leq C^{1/m} n^c$. For $m \to \infty$ we have $C^{1/m} \to 1$. So indeed

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \leq n^c\right) . \tag{4}$$

We similarly derive the converse inequality $\|n\| \geq n^c$, $n \in \mathbb{N}_0$. For every $n \in \mathbb{N}$ the above expansion of $n$ in base $n_0$ gives that

$$n_0^{s+1} > n \geq n_0^s \ .$$

By the $\Delta$-inequality,

$$\|n_0\|^{s+1} = \|n_0^{s+1}\| \leq \|n\| + \|n_0^{s+1} - n\| \ .$$

Hence

$$\begin{aligned}
\|n\| \quad &\geq \quad \|n_0\|^{s+1} - \|n_0^{s+1} - n\| \overset{(2) \text{ and } (4)}{\geq} n_0^{(s+1)c} - (n_0^{s+1} - n)^c \\
&\overset{n \geq n_0^s}{\geq} \quad n_0^{(s+1)c} - (n_0^{s+1} - n_0^s)^c = n_0^{(s+1)c}\left(1 - \left(1 - \tfrac{1}{n_0}\right)^c\right) \\
&\overset{n_0^{s+1} > n}{\geq} \quad n^c C' \text{ where } C' = 1 - \left(1 - \tfrac{1}{n_0}\right)^c > 0 \ .
\end{aligned}$$

The trick with the $m$-th root gives again

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \geq n^c\right) \ .$$

Hence

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| = n^c\right) \ .$$

From multiplicativity of the norm $\|\cdot\|$ we get that $\|x\| = |x|^c$ for any $x \in \mathbb{Q}$. By Exercise 2, $c \in (0, 1]$. Thus case 2 of Ostrowski's theorem holds.

2. $\forall\, n \in \mathbb{N}$ *one has* $\|n\| \leq 1$ *and* $\exists\, n \in \mathbb{N}$ *with* $\|n\| < 1$. Let $n_0$ be minimum such $n$; again $n_0 \geq 2$. We claim that $n_0 = p$ is a prime number. Indeed, if we could express $n_0 = n_1 n_2$ with $n_i \in \mathbb{Z}$ and $1 < n_1, n_2 < n_0$, the contradiction

$$1 > \|n_0\| = \|n_1 n_2\| = \|n_1\| \cdot \|n_2\| = 1 \cdot 1 = 1$$

follows (we used multiplikativity of norms and that $\|m\| = 1$ for any $m \in \mathbb{N}$ with $1 \leq m < n_0$). We show that every prime number $q$ with $q \neq p$ has the norm $\|q\| = 1$. For the contrary let $q \neq p$ be another prime number with $\|q\| < 1$. We take a large $m \in \mathbb{N}$ such that $\|p\|^m, \|q\|^m < \frac{1}{2}$. By Exercise 5 there are integers $a$ and $b$ such that $aq^m + bp^m = 1$. Taking norms we get that

$$1 = \|1\| = \|aq^m + bp^m\| \le \|a\| \cdot \|q\|^m + \|b\| \cdot \|p\|^m < 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = 1 \ .$$

which is a contradiction; we used the triangle inequality, multiplicativity of norms, and the fact that now $\|a\| \le 1$ for every $a \in \mathbb{Z}$.

Thus $\|q\| = 1$ for every prime number $q \ne p$. From this, multiplicativity of norms and prime factorization of any non-zero fraction $x$ we get the expression

$$\begin{aligned} \|x\| \ &= \ \Big\| \prod_{q=2,\,3,\,5,\,\dots} q^{\mathrm{ord}_q(x)} \Big\| = \prod_{q=2,\,3,\,5,\,\dots} \|q\|^{\mathrm{ord}_q(x)} = \|p\|^{\mathrm{ord}_p(x)} \\ &= \ c^{\mathrm{ord}_p(x)}, \ \text{where } c = \|p\| \in (0, 1) \ . \end{aligned}$$

Also $\|0\| = c^{\mathrm{ord}_p(0)} = c^{\infty} = 0$. We are in case 3 of Ostrowski's theorem. □

The preceding proof is taken from the book by Neal Koblitz,

*p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Springer-Verlag, New York, 1984.

This book contains lot of information about the $p$-adic norm $\| \cdot \|_p$ and related $p$-adic analysis.

• *Compact sets in metric spaces.* We review limits of sequences in MSs. Let $(M, d)$ be a MS, $(a_n) \subset M$ be a sequence of points in it and $a \in M$ be a point. We say that $(a_n)$ has the *limit $a$ (in $(M, d)$)* if

$$\forall \varepsilon \, \exists n_0 \left( n \ge n_0 \Rightarrow d(a_n, a) < \varepsilon \right) \ .$$

From now on $\varepsilon > 0$ is a real number and $n_0, n \in \mathbb{N}$. We write that $\lim a_n = a$ or $\lim_{n\to\infty} a_n = a$. If the sequence $(a_n)$ has a limit, we say that it is *convergent*, otherwise it is *divergent*.

Let $(M, d)$ be a MS and $X \subset M$, for example $X = M$. We say that the set $X$ is *compact* if

$$\forall\,(a_n) \subset X\ \exists\,(a_{m_n})\ \exists\,a \in X\ \big(\lim_{n\to\infty} a_{m_n} = a\big)\ .$$

In words: every sequence of points in the set $X$ has a convergent subsequence with limit in $X$. The MS $(M, d)$ *is compact* when the set $M$ is compact.

The Bolzano–Weierstrass theorem states that on the real axis, i.e., in the MS $(\mathbb{R}, |x - y|)$, every closed and bounded interval $X = [a, b]$ is a compact set. We give a few examples of compact sets and compact MSs.

**Exercise 7** *In every* MS *every finite set is compact.*

**Exercise 8** *Is the real axis (with the metric* $|x-y|$*) a compact* MS*?*

**Exercise 9** *Which other intervals on the real axis besides* $[a, b]$ *are compact sets?*

**Exercise 10** *Let* $X = [a, b] \times [c, d]$ *be a rectangle in the plane, that is, in the Euclidean space* $(\mathbb{R}^2, e_2)$*. Prove that* $X$ *is compact.*

**Exercise 11** *Let* $(M, d)$ *be* MS *and* $A, B \subset M$*. Which of the* *following implications holds?*

$$A \text{ and } B \text{ are compact} \Rightarrow A \cup B \text{ is compact}$$
$$A \text{ and } B \text{ are compact} \Rightarrow A \cap B \text{ is compact}$$
$$A \subset B \text{ and } B \text{ is compact} \Rightarrow A \text{ is compact}$$
$$A \text{ and } B \text{ are compact} \Rightarrow A \setminus B \text{ is compact}$$

• *Extrema and compact sets.* We begin with continuous maps between MSs. Let $(M, d)$ and $(N, e)$ be MSs and $f\colon M \to N$ be a map between them. We say that it is *continuous in the point* $a \in M$ if

$$\forall\, \varepsilon\; \exists\, \delta\; \forall\, x \in M\; \big(d(x,\, a) < \delta \Rightarrow e(f(x),\, f(a)) < \varepsilon\big)\;.$$

Here $\delta > 0$ is a real number. *A map $f$ is continuous* if it is continuous in every point $a \in M$.

**Exercise 12** *Let $f\colon M \to N$ be a map between* MSs *and $a \in M$ be a point. Prove Heine's definition of continuity:*

$$f \text{ is continuous in } a \iff$$
$$\iff \forall\, (a_n) \subset M\; \big(\lim\, a_n = a \Rightarrow \lim\, f(a_n) = f(a)\big)\;.$$

**Theorem 13 (attaining extrema)** *Let $(M, d)$ be a* MS*,*

$$f\colon M \to \mathbb{R}$$

*be a continuous function from $M$ to the real axis, and $X \subset M$ be a nonempty compact set. Then*

$$\exists\, a,\, b \in X\; \forall\, x \in X\; \big(f(a) \le f(x) \le f(b)\big)\;.$$

*Thus $f$ attains on the set $X$ both the smallest value $f(a)$ and the largest value $f(b)$.*

**Proof.** First we show that $f[X] = \{f(x) \mid x \in X\}$ is a bounded subset of $\mathbb{R}$. If $f[X]$ were not bounded from above, we could take a sequence $(a_n) \subset X$ with $\lim f(a_n) = +\infty$, i.e., such that $\forall c\ \exists n_0 \big(n \geq n_0 \Rightarrow f(a_n) > c\big)$. By the assumption, $(a_n)$ has a convergent subsequence $(a_{m_n})$ with $\lim a_{m_n} = a \in X$. By the continuity of $f$ in $a$ and Exercise 12, $\lim f(a_{m_n}) = f(a) \in \mathbb{R}$. But this is a contradiction because $\lim f(a_{m_n}) = +\infty$. Boundedness of $f[X]$ from below follows in a similar way.

Thus we define the real numbers $A = \inf(f[X])$ and $B = \sup(f[X])$. By the definition of infima, there is a sequence $(a_n) \subset X$ such that $\lim f(a_n) = A$. By the assumption, $(a_n)$ has a convergent subsequence $(a_{m_n})$ with $\lim a_{m_n} = a \in X$. By the continuity of $f$ in $a$ and Exercise 12, $\lim f(a_{m_n}) = f(a)$. Since subsequences preserve limits, $\lim f(a_{m_n}) = A$. Thus $f(a) = A$ and for every $x \in X$,

$$f(a) = A \leq f(x)$$

because $A = \inf(f[X])$. We produce the element $b \in X$ in a similar way. $\Box$

• *Products of metric spaces.* For the MSs $(M, d)$ and $(N, e)$, we define their *product* $(M \times N, d \times e)$ so that $M \times N$ is the Cartesian product of the sets $M$ and $N$ and the $d \times e$ metric on it is given by

$$(d \times e)((a_1,\, a_2),\, (b_1,\, b_2)) = \sqrt{d(a_1,\, b_1)^2 + e(a_2,\, b_2)^2}\ .$$

**Exercise 14** *Prove that the product of two* MSs *is a* MS.

**Exercise 15** *Prove that the product of two Euclidean* MSs

$$(\mathbb{R}^m,\, e_m) \quad \textit{and} \quad (\mathbb{R}^n,\, e_n)$$

*is (except for a formality in notation) the Euclidean* MS

$$(\mathbb{R}^{m+n},\, e_{m+n}) \ .$$

*What is the "formality"?*

• *Characterization of compact sets in Euclidean* MSs. We defined the ball $B(a,r)$ in a MS last time. A set $X \subset M$ in a MS $(M,d)$ is *open* if

$$\forall\, a \in X\ \exists\, r\ \big(B(a,r) \subset X\big) \ .$$

Here $r > 0$ is a real number, the radius of the ball $B(a,r)$. $X$ is *closed* if $M \setminus X$ is open. $X$ is *bounded* if

$$\exists\, a \in M\ \exists\, r\ \big(X \subset B(a,\, r)\big) \ .$$

The *diameter* of the set $X$ is, for $V = \{d(a,b) \mid a,b \in X\} \subset [0,+\infty)$, defined as

$$\mathrm{diam}(X) := \begin{cases} \sup(V) & \ldots \ \text{the set } V \text{ is bounded from above and} \\ +\infty & \ldots \ \text{the set } V \text{ is unbounded from above .} \end{cases}$$

**Exercise 16** *Prove that any set $X$ is bounded if and only if* $\mathrm{diam}(X) < +\infty$.

**Exercise 17** *Prove that for any unbounded set $X$ there is a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) > 1$.*

In the following two exercises we review basic properties of open and closed sets in a MS.

**Exercise 18** *Let $(M,d)$ be a* MS. *Then the following holds.*

1. *The sets $\emptyset$ and $M$ are both open and closed.*

2. *Any finite intersection of open subsets of $M$ is an open set and any finite union of closed subsets of $M$ is a closed set.*

3. *Any union of open subsets of $M$ is an open set and any intersection of closed subsets of $M$ is a closed set.*

**Exercise 19** *Let $(M,d)$ be a* MS *and $X \subset M$. Then*

$$\text{the set } X \text{ is closed} \iff$$
$$\iff \forall\,(a_n) \subset X\ \forall\, a \in M\ \big(\lim a_n = a \Rightarrow a \in X\big)\ .$$

**Theorem 20 (on compactness)** *The following holds.*

1. *If $X \subset M$ is a compact set in a* MS $(M,d)$*, then $X$ is closed and bounded. The opposite implication does not in general hold, by Exercise 22.*

2. *If $(M,d)$ and $(N,e)$ are two compact* MSs*, then their product $(M \times N, d \times e)$ is a compact* MS.

**Proof.** 1. If $X$ is not closed, then by Exercise 19 there exists a convergent sequence $(a_n) \subset X$ such that $\lim a_n = a \in M \setminus X$. This sequence does not have a convergent subsequence with limit in $X$, since each subsequence has limit $a$. When $X$ is not bounded, we easily construct a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) > 1$ (Exercise 17). This sequence clearly has no convergent subsequence.

2. Let $(a_n) = ((a_{n,1}, a_{n,2}))$ be a sequence in the product MS. We choose a subsequence $(b_n)$ such that $(b_{n,1})$ has a limit $b \in M$ in $(M,d)$. From $(b_n)$ we select a subsequence $(c_n)$ such that $(c_{n,2})$ has a limit $c \in N$ in $(N,e)$. It is not difficult to see that $(c_n)$ is

a subsequence of the sequence $(a_n)$ and that it has in the product MS the limit

$$\lim c_n = (b,\, c) \in M \times N\ .$$

$\square$

**Exercise 21** *Let* $(M, d)$ *be a compact* MS *and* $X \subset M$ *be a closed set. Prove that* $X$ *is compact.*

**Exercise 22** *Let* $M$ *be an infinite set and the metric* $d$ *on it is given as* $d(a, b) = 1$ *for* $a \neq b$ *and* $d(a, a) = 0$. *Show that* $(M, d)$ *is a* MS *that is bounded and closed but not compact.*

**Theorem 23 (compact sets in $\mathbb{R}^n$)** *In every Euclidean* MS $(\mathbb{R}^n, e_n)$, $X \subset \mathbb{R}^n$ *is compact if and only if it is bounded and closed.*

**Proof.** By the first part of the previous theorem, it suffices to prove that every bounded and closed set $X \subset \mathbb{R}^n$ is compact. From its boundedness it follows that for a real number $a > 0$,

$$X \subset K = [-a,\, a]^n = [-a,\, a] \times [-a,\, a] \times \cdots \times [-a,\, a] \subset \mathbb{R}^n\ .$$

The Euclidean MS $(K, e_n)$ is compact by the Bolzano–Weierstrass theorem, part 2 of the previous theorem, and Exercise 15. Clearly, $X$ is also closed in $(K, e_n)$ (problem 24), so according to Exercise 21, $X$ is compact in $(K, e_n)$ and therefore in $(\mathbb{R}^n, e_n)$ (Exercise 25). $\square$

**Exercise 24** *Let* $(M, d)$ *be a* MS, $A \subset B \subset M$ *and* $A$ *be a closed set in* $(M, d) \Rightarrow A$ *is closed also in the subspace* $(B, d)$.

**Exercise 25** *Let* $(M, d)$ *be a* MS *and* $A \subset B \subset M$. *Then* $A$ *is compact in* $(M, d) \iff A$ *is compact in the subspace* $(B, d)$.

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 5, 9, 11, 17 and 22.