MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
lecturer: Martin Klazar

LECTURE 1 (February 15, 2023) METRIC SPACES. HEMISPHERE IS NOT FLAT. p-ADIC ULTRAMETRICS.

Syllabus in SIS:

1. Metric spaces: completeness, connectivity, compactness
2. Series: series of real/complex numbers, power series and series of functions. Different types of convergence, operations with series. Fourier series
3. Complex analysis: holomorphic functions, Cauchy formula - poles of a function, applications
4. Introduction to differential equations: equations with separated variables, linear equations. Existence of a solution, numerical view

We will follow it approximately.

A *metric space* (briefly MS) is a pair (M, d) of a set $M \neq \emptyset$ and a map

$$d\colon M \times M \to \mathbb{R}\ ,$$

so called *metric* or *distance*. Moreover, for every $x, y, z \in M$ it is true that

1. $d(x,\, y) = 0 \iff x = y$.
2. $d(x,\, y) = d(y,\, x)$ (symmetry).

3. $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

Exercise 1. *Show that always* $d(x,y) \geq 0$.

Metric spaces with infinite distances are also considered but here we will not use them.

Every subset $X \subset M$ determines a new MS (X, d'), a *subspace* of (M, d): for $x, y \in X$ we set $d'(x, y) := d(x, y)$. Both metrics are usually denoted by the same symbol and we have a MS (X, d). An *isometry* f of MSs (M, d) and (N, e) is a bijection $f\colon M \to N$ preserving distances:

$$\forall\, x,\, y \in M \, \big(d(x,\, y) = e(f(x),\, f(y))\big).$$

If it exists, we say that the spaces (M, d) and (N, e) are *isometric*. It means that they are practically indistinguishable.

The most important example of a MS is the (n-dimensional) *Euclidean space* $(\mathbb{R}^n, e_n)$, $n \in \mathbb{N} = \{1, 2, \dots\}$, with the metric e_n given for $\overline{x} = (x_1, \dots, x_n)$ and $\overline{y} = (y_1, \dots, y_n)$ in $\mathbb{R}^n$ by the formula

$$e_n(\overline{x},\, \overline{y}) := \sqrt{\textstyle\sum_{i=1}^n (x_i - y_i)^2}\ .$$

Geometrically, e_n is the length of the segment joining the points $\overline{x}$ and $\overline{y}$. By an *Euclidean space* we mean more generally every subspace (X, e_n) when $X \subset \mathbb{R}^n$.

Proposition 2 ($\mathbb{R}^n$ is a MS). $(\mathbb{R}^n, e_n)$ *is a metric space.*

Proof. Clearly, the function e_n has properties 1 and 2 of a metric. For $n = 1$, the triangle inequality is trivial, and for $n \geq 2$ we can prove it geometrically by reducing it to planar case $n = 2$. Indeed, three different non-collinear points in $\mathbb{R}^n$ determine a unique two-dimensional plane $R \subset \mathbb{R}^n$, and all three distances between them in $\mathbb{R}^n$ are the same as in R. This is a non-trivial geometric property of the Euclidean distance, and you can prove a generalization of it in Exercise 4. So it suffices to prove the triangle inequality in $\mathbb{R}^2$. Let $A, B, C \in \mathbb{R}^2$ be three different non-collinear points (otherwise the triangle inequality holds for them trivially) and assume that

$$e_2(A, B) \geq e_2(A,\, C),\; e_2(C,\, B)\;.$$

It suffices to prove that $e_2(A, B) \leq e_2(A, C) + e_2(C, B)$. According to Exercise 3, the heel D of the height from C to the line AB lies inside the segment AB. We consider two right-angled triangles ADC and BDC, with a right angle at vertex D. Then, using the Exercise 3 twice, we have

$$e_2(A,\, B) = e_2(A,\, D) + e_2(D,\, B) < e_2(A,\, C) + e_2(C,\, B)\;.$$

$\square$

Exercise 3. *Let $ABC \subset \mathbb{R}^2$ be a right triangle with a right angle at the vertex B. Then $e_2(A, B), e_2(B, C) < e_2(A, C)$.*

Exercise 4. *Let $m \leq n$ be natural numbers and $R \subset \mathbb{R}^n$ be an affine subspace in $\mathbb{R}^n$ of dimension m, e.g. for $m = 1$ R is a straight line and for $m = 2$, R is a plane. Show that the* MS*s* (R, e_n) *and* $(\mathbb{R}^m, e_m)$ *are isometric.*

Exercise 5. *Let $G = (V, E)$ be a connected (and not necessarily finite) graph and let a function $d\colon V \times V \to \mathbb{N}_0$ $(= \{0, 1, 2, \dots\})$ be defined as*

$$d(u,\, v) := \#\ \textit{edges on the shortest path in } G \textit{ from } u \textit{ to } v\;.$$

Decide whether (V, d) is a MS.

Exercise 6. *Let* $M = \mathcal{R}(a,b)$ *be the set of functions that have a Riemann integral on the interval* $[a,b] \subset \mathbb{R}$*, and for* $f, g \in M$ *let*

$$d(f,\, g) := \int_a^b |f(t) - g(t)| \,\mathrm{dt}\ .$$

Decide whether (M, d) *is a* MS.

Exercise 7 *Let* $A \neq \emptyset$ *be a set (an alphabet) and*

$$M = A^n := \{u = u_1 u_2 \dots u_n \mid u_i \in A\},\ \ n \in \mathbb{N}\ ,$$

be the set of words over the alphabet A *of length* n*. We define the function* $d\colon M \times M \to \mathbb{N}_0$ *as*

$$d(u,\, v) := \#\{i \in \{1, 2, \dots, n\} \mid u_i \neq v_i\}\ .$$

Prove that (M, d) *is a* MS.

Exercise 8. *Let* $X \neq \emptyset$ *be a set and let*

$$M := \{f \mid (f\colon X \to \mathbb{R}) \wedge \exists\, c > 0 \big(x \in X \Rightarrow |f(x)| < c\big)\}$$

be the set of bounded real functions defined on X*. For* $f, g \in M$ *we define*

$$d(f,\, g) := \sup(\{|f(x) - g(x)| \mid x \in X\})\ .$$

Prove that (M, d) *is a* MS.

Spherical metric. We denote by

$$S := \{(x_1,\, x_2,\, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$$

the *unit sphere* (a sphere with radius 1) in the Euclidean space $\mathbb{R}^3$. The function $s\colon S \times S \to [0, \pi]$ is defined for $\overline{x}, \overline{y} \in S$ as

$$s(\overline{x},\, \overline{y}) = \begin{cases} 0 & \dots \quad \overline{x} = \overline{y} \ \text{ and} \\ \varphi & \dots \quad \overline{x} \neq \overline{y}\ , \end{cases}$$

where φ is the angle subtended by two straight lines passing through the *origin* $\overline{0} := (0,0,0)$ and the points $\overline{x}$ and $\overline{y}$, respectively. This angle is actually the length of the shorter of the arcs between the points $\overline{x}$ and $\overline{y}$ on the unit circle cut on S by the plane determined by the points $\overline{0}$, $\overline{x}$ and $\overline{y}$. We call the function s the *spherical metric.*

To prove that s is a metric, we introduce the following terminology. For $A \in \mathbb{R}^3$, an *A-ray* is an half-line in $\mathbb{R}^3$ starting in A, and $u(p,q) \in (0,\pi)$ is the angle between two different and non-antipodal A-rays p and q. For an A-ray q and $\beta \in (0,\pi)$ we mean by the *cone* $K = K(q,\beta)$ (with the axis q and vertex A) the surface

$$K := \bigcup\{p \mid p \text{ is an } A\text{-ray and } u(p,\,q) = \beta\}\ .$$

Proposition 9 (S is a MS). *(S,s) is a metric space.*

Proof. Clearly, the function s has the properties 1 and 2 of a metric. It suffices to prove property 3 for any three points on S. We assume that they are mutually different and that no two of them are antipodal (symmetric about the origin), for else the triangle inequality holds for them trivially. We prove that if p, q and r are three different and non-antipodal $\overline{0}$-rays (determined by the given points), $u(p,q) = \alpha$, $u(q,r) = \beta$ and $u(p,r) = \gamma$, then $\gamma \le \alpha + \beta$. We assume that p, q, α and β are given and find $r \subset K(q,\beta)$ such that the angle γ is maximized. By Exercise 10, r is equal to one of the two $\overline{0}$-rays forming the intersection of $R \cap K(q,\beta)$, where R is the plane determined by p and q. In R we are in a planar situation and it is clear that for three different and non-antipodal half-lines in $\mathbb{R}^2$ starting from the origin $(0,0)$ and the angles $\alpha,\beta,\gamma \in (0,\pi)$ determined by their pairs the inequality $\gamma \le \alpha + \beta$ always holds

(Exercise 11). $\square$

Exercise 10. *Let p and q be two different non-antipodal $\overline{0}$-rays, $\beta \in (0, \pi)$ and $R \subset \mathbb{R}^3$ be the plane determined by p and q. Then the largest angle*

$$\max_{r \subset K(q,\,\beta)} u(p,\, r)$$

is attained on one of the $\overline{0}$-rays forming $K(q, \beta) \cap R$.

Exercise 11. *Let $p, q, r \subset \mathbb{R}^2$ be three half-lines that are different, non-antipodal and start from the origin $(0, 0)$ and $\alpha, \beta, \gamma \in (0, \pi)$ be the angles determined by their pairs. Then $\gamma \leq \alpha + \beta$.*

We prove that the spherical metric differs substantially from the Euclidean metric.

The (upper) *hemisphere* H is the set

$$H := \{(x_1,\, x_2,\, x_3) \in S \mid x_3 \geq 0\} \subset S\ .$$

Theorem 12 (H is not flat). *The metric space (H, s) is not isometric to any Euclidean space (X, e_n) with $X \subset \mathbb{R}^n$.*

Proof. The following property of distances between four points t, u, v and w in the Euclidean space $(\mathbb{R}^n, e_n)$ is not satisfied in (H, s):

$$\begin{aligned} & e_n(t,\, u) = e_n(t,\, v) = e_n(u,\, v) > 0 \wedge \\ & \wedge\, e_n(t,\, w) = e_n(w,\, u) = \tfrac{1}{2} e_n(t,\, u) \Rightarrow \\ & \Rightarrow\ e_n(w,\, v) = \tfrac{\sqrt{3}}{2} e_n(t,\, v)\ (< e_n(t,\, v))\ . \end{aligned}$$

According to the assumption of implication, the points t, u and v form an equilateral triangle with a side of length $x > 0$, and w has from t and from u distance $\frac{x}{2}$. According to Exercise 13, then

w is the center of the segment tu. These four points are therefore coplanar (they all lie in the same plane) and the line segment vw is the height dropped from the vertex v of the equilateral triangle tuv to the side tu. According to the Pythagorean theorem, its length $e_2(v, w) = e_n(v, w)$ (see Exercise 4) equals to $\frac{\sqrt{3}}{2}x$, which is exactly the conclusion of the implication.

We find on the hemisphere (H, s) four different points t, u, v and w satisfying the assumption of the previous implication, but not its conclusion. It follows that the isometry between the hemisphere and the Euclidean space does not exist, because every isometry preserves by its definition the implication. These points are

$$t = (1,\,0,\,0),\;\; u = (0,\,1,\,0),\;\; v = (0,\,0,\,1) \text{ a } w = \left(\tfrac{1}{\sqrt{2}},\,\tfrac{1}{\sqrt{2}},\,0\right)\;.$$

Clearly, $s(t, u) = s(t, v) = s(u, v) = \frac{\pi}{2}$ and $s(t, w) = s(w, u) = \frac{1}{2}s(t, u) = \frac{\pi}{4}$. Point v is the „north pole“ ($x_3 = 1$), t, u and w lie on the “equator” ($x_3 = 0$) and w is the center of the arc tu. But all points on the equator have the same distance $\frac{\pi}{2}$ from the pole v. So $s(w, v) = s(t, v)$ and the conclusion of the implication does not hold. $\square$

Could not the previous proof be simplified so that we only argue with three-point configurations (Exercise 15)? And what if instead of the whole hemisphere we take only a small spherical cap (Exercise 16)?

Exercise 13. *When $a, b, c \in \mathbb{R}^n$ are different points in an Euclidean space with distances $e_n(c, a) = e_n(c, b) = \frac{1}{2}e_n(a, b)$, then c is the midpoint of the segment ab.*

Exercise 14. *Does the analogy of the previous result hold in* *the* MS (S, s)*?*

Exercise 15. *Can any spherical triangle be isometrically realized in the Euclidean plane* $(\mathbb{R}^2, e_2)$*?*

Exercise 16. *Prove that no spherical cap (portion of the sphere* S *cut off by a plane) with spherical metric is isometric to the Euclidean space* (X, e_n)*.*

The metric d in a MS (M, d) is called an *ultrametric*, or a *non-Archimedean metric*, if it satisfies the *strong triangle inequality*

$$\forall\, x,\, y,\, z \in M\ \big(d(x,\, y) \leq \max(\{d(x,\, z),\, d(z,\, y)\})\big)\ .$$

Since $\max(d(x, z), d(z, y)) \leq d(x, z) + d(z, y)$, every ultrametric is a metric. In the following proposition and exercise, we will see that when working in ultrametric spaces, abbreviated UMS, one cannot rely on the intuition developed in Euclidean spaces.

Proposition 17 (triangles in UMS). *In the ultrametric space* (M, d)*, every triangle is isosceles, that is, it has two sides of equal length.*

Proof. Let $x, y, z \in M$ be three different points in the UMS (M, d). Let $d(x, y) \geq d(x, z), d(z, y)$. Since d is an ultrametric, it follows from

$$d(x,\, y) \leq \max(\{d(x,\, z),\, d(z,\, y)\})$$

that $d(x, y) = d(x, z)$ or $d(x, y) = d(z, y)$. $\square$

An (open) *ball* (with center $a \in M$ and radius $r > 0$) in a MS (M, d) is the subset

$$B(a,\, r) := \{x \in M \mid d(x,\, a) < r\} \subset M\ .$$

Always $B(a,r) \neq \emptyset$ because $a \in B(a,r)$.

Exercise 18. *Prove that for every ball in a* UMS*, every point of it is its center.*

Exercise 19. *Prove an important supplement to the strong triangle inequality: if $d(x,z) \neq d(z,y)$ then the equality holds.*

UMSs may appear bizarre when first encountered, but the strong triangle inequality actually simplifies many things. For example – unlike in general MSs – infinite series in an UMS converges iff its summand goes to 0. Basic examples of ultrametrics are p-adic distances of fractions, and so we now define them.

p-adic metrics. Let $p \in \{2,3,5,7,11,\dots\}$ be a prime number and let $n \in \mathbb{Z}$ be a nonzero integer. We define the *p-adic order* of the number n as

$$\mathrm{ord}_p(n) := \max(\{m \in \mathbb{N}_0 \mid p^m \,|\, n\}) \ .$$

Here $\cdot \,|\, \cdot$ denotes the *divisibility relation* on $\mathbb{Z}$: for $a,b \in \mathbb{Z}$ one has that

$$a \,|\, b \overset{\text{def}}{\iff} \exists\, c \in \mathbb{Z}\, \big(b = ac\big) \ .$$

For every p we define $\mathrm{ord}_p(0) := +\infty$. We extend the function $\mathrm{ord}_p(\cdot)$ to fractions. For a nonzero fraction $\alpha = \frac{a}{b} \in \mathbb{Q}$, we define

$$\mathrm{ord}_p(\alpha) := \mathrm{ord}_p(a) - \mathrm{ord}_p(b) \ ,$$

and else we put again $\mathrm{ord}_p(0) = \mathrm{ord}_p(0/b) := +\infty$. E.g. we have $\mathrm{ord}_5(297/100) = -2$, $\mathrm{ord}_{11}(297/100) = 1$ and $\mathrm{ord}_3(297/100) = 3$.

Exercise 20. *Show that $\frac{a}{b} = \frac{c}{d} \Rightarrow \mathrm{ord}_p(a/b) = \mathrm{ord}_p(c/d)$.*

Proposition 21 (additivity of $\mathrm{ord}_p(\cdot)$). *It holds that*

$$\forall\, \alpha, \beta \in \mathbb{Q} \left(\mathrm{ord}_p(\alpha\beta) = \mathrm{ord}_p(\alpha) + \mathrm{ord}_p(\beta)\right) .$$

Here $(+\infty) + (+\infty) = (+\infty) + n = n + (+\infty) := +\infty$ *for every* $n \in \mathbb{Z}$.

Proof. Let $\alpha = \frac{a}{b}$ and $\beta = \frac{c}{d}$. The left side of the given equality is then

$$\mathrm{ord}_p(ac) - \mathrm{ord}_p(bd)$$

and the right side is

$$\mathrm{ord}_p(a) - \mathrm{ord}_p(b) + \mathrm{ord}_p(c) - \mathrm{ord}_p(d) .$$

It therefore suffices to prove the above equality for $\alpha, \beta \in \mathbb{Z}$. There it holds due to the *Fundamental Theorem of Arithmetic* (uniqueness of decompositions of numbers into products of powers of prime numbers). $\square$

p-adic norms are an intermediate step to the definition of p-adic metrics. We fix a real constant $c \in (0, 1)$ and define the function $|\cdot|_p \colon \mathbb{Q} \to [0, +\infty)$, so-called *$p$-adic norm*, as

$$\left|\frac{a}{b}\right|_p := c^{\mathrm{ord}_p(a/b)} ,$$

where we put $|0|_p = c^{+\infty} := 0$. E.g. for $c = \frac{1}{2}$, $|\frac{1}{100}|_5 = 4$. It is easy to prove

Exercise 22 (multiplicativity of $|\cdot|_p$). *Prove that for every p, every two fractions α, β, and every $c \in (0, 1)$,*

$$|\alpha \cdot \beta|_p = |\alpha|_p \cdot |\beta|_p .$$

A *normed field* $F = (F, 0_F, 1_F, +_F, \cdot_F, |\cdot|_F)$, abbreviated $(F, |\cdot|_F)$, is any field F equipped with the *norm* $|\cdot|_F\colon F \to [0, +\infty)$, which satisfies the following three requirements.

1. $\forall\, x \in F \left(|x|_F = 0 \iff x = 0_F\right)$.
2. $\forall\, x, y \in F \left(|x \cdot_F y|_F = |x|_F \cdot |y|_F\right)$ (multiplicativity).
3. $\forall\, x, y \in F \left(|x +_F y|_F \le |x|_F + |y|_F\right)$ (triangle inequality).

Basic examples of normed fields are the field of fractions $\mathbb{Q}$, the field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$, where the norm is the usual absolute value $|\cdot|$.

Exercise 23. *In every normed field $(F, |\cdot|_F)$, $|1_F|_F = 1$. For every $x \in F$, $|-x|_F = |x|_F$. For every $x \ne 0_F$, $|x^{-1}|_F = 1/|x|_F$.*

Exercise 24. *Prove that for every normed field $(F, |\cdot|_F)$ the function*

$$d(x,\, y) := |x - y|_F$$

is a metric on F. Prove that when $|\cdot|_F$ satisfies a strong triangle inequality (defined in an obvious way, see below), then d is an ultrametric.

Other important examples of normed fields are provided by the field of fractions $\mathbb{Q}$ equipped with p-adic norms.

Proposition 25 (on $|\cdot|_p$). *For every prime p and constant $c \in (0,1)$, $(\mathbb{Q}, |\cdot|_p)$ is a normed field. The corresponding metric space $(\mathbb{Q}, d)$, defined by Exercise 24, is an ultrametric space.*

Proof. The multiplicativity of the norm $|\cdot|_p$ is proved in Exercise 22. It remains to prove the strong triangle inequality

$$\forall\, \alpha,\, \beta \in \mathbb{Q} \left(|\alpha + \beta|_p \le \max(\{|\alpha|_p,\, |\beta|_p\})\right).$$

This is equivalent to the inequality for p-adic orders

$$\forall\, \alpha,\, \beta \in \mathbb{Q} \left(\mathrm{ord}_p(\alpha+\beta) \geq \min(\mathrm{ord}_p(\alpha),\, \mathrm{ord}_p(\beta))\right) .$$

Let $\alpha = \frac{a}{b}$ and $\beta = \frac{c}{d}$ be nonzero fractions (when $\alpha = 0$ or $\beta = 0$, the inequality holds trivially as an equality) and $\mathrm{ord}_p(\alpha) =: m$ and $\mathrm{ord}_p(\beta) =: n$ are in $\mathbb{Z}$. In other words,

$$\frac{a}{b} = p^m \cdot \frac{a'}{b'} \text{ and } \frac{c}{d} = p^n \cdot \frac{c'}{d'} ,$$

where none of the integers a', b', c' and d' is divisible by p. Let us say that $m \leq n$. Then

$$\frac{a}{b} + \frac{c}{d} = p^m \cdot \frac{a'd' + p^{n-m}c'b'}{b'd'} =: p^m \cdot \frac{e}{f} .$$

Clearly, $e, f \in \mathbb{Z}$ and p does not divide f, so $\mathrm{ord}_p(e/f) \geq 0$. Due to the additivity of p-adic order,

$$\mathrm{ord}_p(\alpha+\beta) \geq m = \min(\mathrm{ord}_p(\alpha), \mathrm{ord}_p(\beta)) .$$

$\square$

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 3, 5, 6, 19 and 20.