

MATHEMATICAL ANALYSIS 3 (NMAI056)

summer term 2025/26

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LECTURE 7 (March 30, 2026)

THE BASEL PROBLEM AND FOURIER SERIES

- *The Basel problem.* What is the sum

$$B = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots?$$

According to Wikipedia, this problem was posed by Pietro Mengoli in 1650. In 1734, Leonard Euler solved it:

$$B = \frac{\pi^2}{6}.$$

The problem is named after Euler's hometown Basel. The mathematical clan of the Bernoullis resided there; they also tried to solve the Basel problem, but they did not succeed. (Do you know where the Basel square is in Prague and what shape it has?)

- *Series.* We review basic notions of the theory of (infinite) series to make sense of the previous problem. A *series* $\sum a_n = \sum_{n=1}^{\infty} a_n$ is actually a sequence $(a_n) \subset \mathbb{R}$, to which we assign the sequence of *partial sums*

$$(s_n) = (a_1 + a_2 + \dots + a_n) \subset \mathbb{R}.$$

The *sum* of the series is $\lim s_n$, if this limit exists. If this limit is finite ($\in \mathbb{R}$), the series *converges*, else (the sum is $\pm\infty$ or does not exist) it *diverges*. Sums of series are denoted by the same symbols as the series themselves,

$$\sum a_n = \sum_{n=1}^{\infty} a_n = \lim s_n = \lim(a_1 + a_2 + \dots + a_n).$$

We review basic results about series in a couple of exercises.

Exercise 1 If the series $\sum a_n$ converges then $\lim a_n = 0$.

Exercise 2 If the series $\sum a_n$ has almost all summands non-negative, i.e. $n \geq n_0 \Rightarrow a_n \geq 0$, then $\sum a_n$ converges or has the sum $+\infty$.

Exercise 3 The sum $\sum \frac{1}{n} = +\infty$.

Exercise 4 The sum $\sum \frac{1}{(n+1)n} = 1$.

Exercise 5 Using the previous exercise, prove that the series $\sum n^{-2}$ in the Basel problem converges.

Exercise 6 For $q \in (-1, 1)$, the sum

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}.$$

Exercise 7 If $a_1 \geq a_2 \geq \dots \geq 0$ and $\lim a_n = 0$, then the series $\sum (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \dots$ converges.

Exercise 8 Prove simply that if the sum

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \text{ then the sum } \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

• *Riemannian series.* We call a series $\sum a_n$ *Riemannian* if (i) $\lim a_n = 0$, (ii) $\sum a_{k_n} = +\infty$ and (iii) $\sum a_{z_n} = -\infty$, where (a_{k_n}) , resp. (a_{z_n}) , is the subsequence of nonnegative, resp. negative, summands in the series $\sum a_n$.

Exercise 9 Fill in details in the sketch of the next proof.

Theorem 10 (Riemann) Let $\sum a_n$ be a Riemannian series. Then for every $S \in \mathbb{R}^*$ there is a bijection $\pi: \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S$$

— by reordering a Riemannian series we can get any sum.

Proof. Suppose that $\sum a_n$ is a Riemannian series and that $\sum a_{k_n}$ and $\sum a_{z_n}$ are as in the definition. We define π for any given $S \in \mathbb{R}$ (i.e., S is a real number, not $\pm\infty$) as follows. We initialize three variables by $i = 1$, $j = 0$ and $\pi(1) = k_1$. Suppose that $\pi(1), \pi(2), \dots, \pi(n)$ have been already defined and $a = \sum_{k=1}^n a_{\pi(k)}$. If $a < S$ then $i := i + 1$, $j := j$ and $\pi(n + 1) = k_i$. If $a \geq S$ then $i := i$, $j := j + 1$ and $\pi(n + 1) = z_j$. In this way we define a map $\pi: \mathbb{N} \rightarrow \mathbb{N}$. It follows that π is a bijection and that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S.$$

□

• *Trigonometric series.* These are the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where $a_n, b_n \in \mathbb{R}$ are the *coefficients* and $x \in \mathbb{R}$. Our goal is to express a wide class of functions $f: [-\pi, \pi] \rightarrow \mathbb{R}$ as trigonometric series. We use these expressions to solve the Basel problem.

Let $\mathcal{R}(-\pi, \pi)$ be the set of all Riemann integrable functions $f: [-\pi, \pi] \rightarrow \mathbb{R}$. For $f, g \in \mathcal{R}(-\pi, \pi)$ we define

$$\langle f, g \rangle = \int_{-\pi}^{\pi} fg \quad (\in \mathbb{R})$$

(it follows from the theory of the Riemann \int that $f, g \in \mathcal{R}(-\pi, \pi) \Rightarrow fg \in \mathcal{R}(-\pi, \pi)$).

Exercise 11 *Prove that*

$$\langle f, g \rangle = \langle g, f \rangle, \quad \langle f, f \rangle \geq 0$$

and, for $a, b \in \mathbb{R}$,

$$\langle af + bg, h \rangle = a \langle f, h \rangle + b \langle g, h \rangle .$$

However, it is not a perfect scalar product:

Exercise 12 *The equivalence $\langle f, f \rangle = 0 \iff f = 0$ does not hold.*

A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is 2π -periodic if for every $x \in \mathbb{R}$ we have $f(x + 2\pi) = f(x)$.

Proposition 13 *For every $m, n \in \mathbb{N}_0$ we have*

$$\langle \sin(mx), \cos(nx) \rangle = 0.$$

For every $m, n \in \mathbb{N}_0$, except $m = n = 0$, we have

$$\langle \sin(mx), \sin(nx) \rangle = \langle \cos(mx), \cos(nx) \rangle = \begin{cases} \pi & \dots & m = n & \text{and} \\ 0 & \dots & m \neq n. \end{cases}$$

Finally,

$$\langle \sin(0x), \sin(0x) \rangle = 0 \quad \text{and} \quad \langle \cos(0x), \cos(0x) \rangle = 2\pi.$$

Proof. Let $m, n \in \mathbb{N}_0$. We compute the values

$$S_{m,n} = \langle \sin(mx), \sin(nx) \rangle, \quad T_{m,n} = \langle \cos(mx), \cos(nx) \rangle$$

and

$$U_{m,n} = \langle \sin(mx), \cos(nx) \rangle.$$

Clearly, $S_{0,0} = 0$, $T_{0,0} = 2\pi$ and $U_{0,0} = 0$. Let m or n be non-zero, say $m \neq 0$ (for $n \neq 0$ the calculation is similar). Integration by parts using that $\sin(mx) = (-\cos(mx)/m)'$ and $\cos(mx) = (\sin(mx)/m)'$ yields

$$S_{m,n} = \frac{n}{m} \cdot T_{m,n}, \quad T_{m,n} = \frac{n}{m} \cdot S_{m,n} \quad \text{and} \quad U_{m,n} = -\frac{n}{m} \cdot U_{n,m}$$

– the first term $[\dots]_{-\pi}^{\pi}$ in the formula is always zero because \dots is a 2π -periodic function. The first two equations together give

$$(1 - (n/m)^2)S_{m,n} = 0 = (1 - (n/m)^2)T_{m,n} .$$

If $n \neq m$ then $S_{m,n} = T_{m,n} = 0$. When $n = m$, then we know that $S_{m,m} = T_{m,m}$. But from the identity $\sin^2 x + \cos^2 x = 1$ (holding for every $x \in \mathbb{R}$) it follows that $S_{m,m} + T_{m,m} = \int_{-\pi}^{\pi} 1 = 2\pi$. Thus, $S_{m,m} = T_{m,m} = \pi$. The third equation above for $m = n$ gives $U_{m,m} = -U_{m,m}$ and so $U_{m,m} = 0$. To calculate $U_{m,n}$ for $m \neq n$, we express $U_{n,m}$ by integration by parts again using $\cos(mx) = (\sin(mx)/m)'$:

$$U_{n,m} = -(n/m)U_{m,n} .$$

Together $U_{m,n} = (n/m)^2 U_{m,n}$ and again $U_{m,n} = 0$. In summary: $S_{m,m} = T_{m,m} = \pi$ for $m \in \mathbb{N}$, $S_{0,0} = 0$ and $T_{0,0} = 2\pi$, and all other values of $S_{m,n}$, $T_{m,n}$ and $U_{m,n}$ for $m, n \in \mathbb{N}_0$ are zero. \square

• *The Fourier series of a function.* For $f \in \mathcal{R}(-\pi, \pi)$ we define the *cosine Fourier coefficients* by

$$a_n = \frac{\langle f(x), \cos(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \quad n = 0, 1, \dots$$

and its *sine Fourier coefficients* by

$$b_n = \frac{\langle f(x), \sin(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx, \quad n = 1, 2, \dots$$

The *Fourier series of the function* f ($\in \mathcal{R}(-\pi, \pi)$) is the trigonometric series

$$F_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) ,$$

where a_n and b_n are, respectively, the cosine and sine Fourier coefficients of f . From the perspective of functional analysis, we work in an

infinite-dimensional vector space with the (almost) scalar product $\langle \cdot, \cdot \rangle$, in which the “coordinate axes” (elements of the orthogonal basis) are the functions

$$\{\cos(nx): n \in \mathbb{N}_0\} \cup \{\sin(nx): n \in \mathbb{N}\}.$$

Fourier coefficients of a given function f are the coordinates of f with respect to these infinitely many coordinate axes. In contrast with Cartesian coordinates of points in \mathbb{R}^n , not every function is equal to the sum of its Fourier series. In a moment we present sufficient conditions for this to hold.

• *Bessel’s inequality.*

Theorem 14 (Bessel’s Inequality) *Let $f \in \mathcal{R}(-\pi, \pi)$. Then the Fourier coefficients a_n and b_n satisfy*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \leq \frac{\langle f, f \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2.$$

Proof. We denote by $s_n = s_n(x)$, $n = 1, 2, \dots$, the n -th partial sum of the Fourier series of the function f :

$$\begin{aligned} s_n &= \frac{a_0}{2} + \sum_{k=1}^n (a_k \cos(kx) + b_k \sin(kx)) \\ &= \sum_{k=0}^n (a'_k \cos(kx) + b'_k \sin(kx)), \end{aligned}$$

where

$$a_k = \pi^{-1} \langle f, \cos(kx) \rangle, \quad b_k = \pi^{-1} \langle f, \sin(kx) \rangle, \quad k = 0, 1, 2, \dots,$$

$a'_0 = \frac{a_0}{2}$, $a'_k = a_k$ for $k > 0$, $b'_0 = 0$ and $b'_k = b_k$ for $k > 0$. Due to the linearity of $\langle \cdot, \cdot \rangle$, the definition of a'_k , b'_k , a_k , b_k and the orthogonality of $\sin(kx)$ and $\cos(kx)$, we have

$$\begin{aligned} \langle s_n, s_n \rangle &= \sum_{k=0}^n ((a'_k)^2 \langle \cos(kx), \cos(kx) \rangle + (b'_k)^2 \langle \sin(kx), \sin(kx) \rangle) \\ &= \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) \right). \end{aligned}$$

Also,

$$\begin{aligned}\langle s_n, f \rangle &= \sum_{k=0}^n (a'_k \langle \cos(kx), f \rangle + b'_k \langle \sin(kx), f \rangle) \\ &= \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) \right).\end{aligned}$$

On the other hand,

$$0 \leq \langle f - s_n, f - s_n \rangle = \langle f, f \rangle - 2\langle s_n, f \rangle + \langle s_n, s_n \rangle,$$

hence $2\langle s_n, f \rangle - \langle s_n, s_n \rangle \leq \langle f, f \rangle$. Thus for every n ,

$$\frac{a_0^2}{2} + \sum_{k=1}^n (a_k^2 + b_k^2) = \frac{2\langle s_n, f \rangle - \langle s_n, s_n \rangle}{\pi} \leq \frac{\langle f, f \rangle}{\pi}.$$

The series of squares of the Fourier coefficients of the function f converges and its sum is bounded by the stated value. \square

The next result is called the Riemann–Lebesgue Lemma.

Exercise 15 *Prove that for every function $f \in \mathcal{R}(-\pi, \pi)$.*

$$\lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \lim_{n \rightarrow \infty} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx = 0.$$

Hint: use Bessel's inequality.

• *Piece-wise smooth functions and Dirichlet's theorem.* We call a function $f: [a, b] \rightarrow \mathbb{R}$, where $a < b$ are real numbers, *piece-wise smooth* if there exists a partition

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b, \quad k \geq 1,$$

such that on every interval (a_{i-1}, a_i) , $i = 1, 2, \dots, k$, f has continuous derivative f' , for every $i = 1, 2, \dots, k$ there exist finite one-sided limits

$$f(a_i - 0) := \lim_{x \rightarrow a_i^-} f(x) \quad \text{and} \quad f'(a_i - 0) := \lim_{x \rightarrow a_i^-} f'(x),$$

and for every $i = 0, 1, \dots, k - 1$ there exist finite one-sided limits

$$f(a_i + 0) := \lim_{x \rightarrow a_i^+} f(x) \quad \text{and} \quad f'(a_i + 0) := \lim_{x \rightarrow a_i^+} f'(x).$$

A piece-wise smooth function can be at several points in the interval $[a, b]$ discontinuous, but at the points of discontinuity it has finite one-sided limits and one-sided non-vertical tangents.

Exercise 16 *Is the function $f: [-1, 1] \rightarrow \mathbb{R}$, defined as $f(x) = (-x)^{1/3}$ for $x \in [-1, 0]$ and $f(x) = x^{1/3}$ for $x \in [0, 1]$, piece-wise smooth?*

Exercise 17 *Is the signum function $\text{sgn}: [-1, 1] \rightarrow \mathbb{R}$, defined as $\text{sgn}(x) = -1$ for $x \in [-1, 0)$, $\text{sgn}(0) = 0$ and $\text{sgn}(x) = 1$ for $x \in (0, 1]$, piece-wise smooth?*

Theorem 18 (Dirichlet's) *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a 2π -periodic function such that its restriction to the interval $[-\pi, \pi]$ is piece-wise smooth. Then for every $a \in \mathbb{R}$ its Fourier series $F_f(x)$ sums to*

$$F_f(a) = \frac{f(a+0) + f(a-0)}{2} := \frac{\lim_{x \rightarrow a^+} f(x) + \lim_{x \rightarrow a^-} f(x)}{2}.$$

Thus, at each point of continuity $a \in \mathbb{R}$ of the function $f(x)$, its Fourier series sums to the functional value, $F_f(a) = f(a)$.

Proof. We will probably skip it. □

We say that $f: [a, b] \rightarrow \mathbb{R}$ is *smooth* if f has continuous derivative f' on (a, b) , and the functions $f(x)$ and $f'(x)$ have finite limits at the ends a and b .

Corollary 19 *Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be 2π -periodic and continuous, and let $f|_{[-\pi, \pi]}$ be smooth. Then*

$$F_f(a) = f(a)$$

for every $a \in \mathbb{R}$.

Proof. This follows from the previous theorem: by the assumption f is continuous on \mathbb{R} . \square

So every continuous and smooth function equals to the sum of the Fourier series.

• *Back to Basel problem.* Let $I \subset \mathbb{R}$ be an interval symmetric with respect to the origin and $f: I \rightarrow \mathbb{R}$. We say that f is *even* (resp. *odd*) if $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$) for every $x \in I$.

Exercise 20 *Let $f \in \mathcal{R}(-\pi, \pi)$. Prove that all sine (or cosine) Fourier coefficients of an even (or odd) functions f are zero. How do you simplify cosine (or sine) Fourier coefficients of an even (or odd) function?*

We find the Fourier series of the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined on the interval $[-\pi, \pi]$ by $f(x) = x^2$ and 2π -periodically extended to \mathbb{R} (which is possible as $(-\pi)^2 = \pi^2$). Sine Fourier coefficients are zero according to the previous exercise. The first cosine Fourier coefficient is, according to this exercise,

$$a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \frac{2\pi^2}{3}.$$

Next we have for $n \in \mathbb{N}$ that

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi x^2 \overbrace{\cos(nx)}^{(\sin(nx)/n)'} dx \\
 &= \frac{2}{\pi n} \underbrace{[x^2 \sin(nx)]_0^\pi}_{0-0=0} - \frac{4}{\pi n} \int_0^\pi x \overbrace{\sin(nx)}^{(-\cos(nx)/n)'} dx \\
 &= \frac{4}{\pi n^2} \underbrace{[x \cos(nx)]_0^\pi}_{\pi(-1)^n} - \frac{4}{\pi n^2} \underbrace{\int_0^\pi \cos(nx) dx}_{0-0=0} = (-1)^n \frac{4}{n^2}.
 \end{aligned}$$

Since f is continuous and smooth on $[-\pi, \pi]$, by Corollary 19 we have

$$f(a) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(na) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos(na)}{n^2}$$

for every $a \in \mathbb{R}$. For $a = \pi$ we get

$$\pi^2 = f(\pi) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n^2}, \text{ so that } \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 21 *The function $f(x)$ is defined on $[-\pi, \pi)$ as $f(x) = \pi - x$, and is 2π -periodically extended to \mathbb{R} . Expand it into Fourier series.*

Exercise 22 *What sum of the infinite series do we get from the previous expansion (using Dirichlet's theorem) for $x = \frac{\pi}{2}$?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 8, 9, 16 and 20.