MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
lecturer: Martin Klazar

LECTURE 4 (March 9, 2026) THE PROOF OF FUNDAMENTAL THEOREM OF ALGEBRA. COMPLETE SPACES. BAIRE'S THEOREM

• *n-th complex roots.* To prove the existence of n-th roots in $\mathbb{C}$, we first reduce the situation to odd n and to numbers lying on the complex unit circle $S = \{z \in \mathbb{C}\colon\ |z| = 1\}$.

Exercise 1 *Using the last two exercises in the previous lecture, prove that if for every $u \in S$ and for every odd $n \in \mathbb{N}$ there exists a $v \in S$ such that $v^n = u$, then the following theorem holds.*

Theorem 2 *If $u \in \mathbb{C}$ and $n \in \mathbb{N}$, then there is a number $v \in \mathbb{C}$ such that $v^n = u$.*

Proof. So we can assume that $u \in S$ and that n is odd. We prove that the continuous map $f(z) = z^n\colon S \to S$ is onto. For contradiction, let there be a number $w \in S \setminus f[S]$. Since n is odd, also $-w \in S \setminus f[S]$ (always $f(-z) = -f(z)$). We consider the line $\ell \subset \mathbb{C}$ going through the points w and $-w$. We have the partition

$$\mathbb{C} = A \cup \ell \cup B\ ,$$

where A and B are the open half-planes determined by ℓ. By Exercise 3, A and B are disjoint open sets. By Exercise 4, $(A \cup B) \cap S = S \setminus \{w, -w\}$, $\{1, -1\} \subset f[S] \cap (A \cup B)$ and $|A \cap \{1, -1\}| = 1$. Thus, the sets A and B cut the set $f[S]$ and make it disconnected.

This contradicts Theorem 21 in the last lecture, because $f[S]$ is the image of the connected set S by the continuous function f and is therefore connected. $\square$

Exercise 3 *Prove that for every line $\ell \subset \mathbb{C}$, the complement $\mathbb{C} \setminus \ell$ is a disjoint union of two open sets.*

Exercise 4 *Let $\ell \subset \mathbb{C}$ be a line passing through the origin, $\ell \cap S = \{w, -w\}$ and let A and B be the open half-planes determined by ℓ. Prove that $(A \cup B) \cap S = S \setminus \{w, -w\}$ and that for every $u \in S \setminus \{w, -w\}$, the points u and $-u$ lie in different half-planes A and B.*

We proceed to the second step of the proof of the FTA, which uses compact sets in $\mathbb{C}$. Recall that the complex numbers $\mathbb{C}$ form the metric space $(\mathbb{C}, |u-v|)$ that is isometric to the Euclidean plane $(\mathbb{R}^2, e_2)$.

Exercise 5 *Let $\alpha \le \alpha'$ and $\beta \le \beta'$ be real numbers. Then the rectangle*

$$R = \{a + bi\colon\ \alpha \le a \le \alpha' \wedge \beta \le b \le \beta'\}$$

is a compact set.

Proposition 6 *If Theorem 2 holds, then every non-constant complex polynomial has a root.*

Proof. Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a non-constant complex polynomial: $n \ge 1$, $a_j \in \mathbb{C}$ and $a_n \ne 0$. The function

$$f(z) = |p(z)|\colon \mathbb{C} \to [0, +\infty) \ \ (\subset \mathbb{C})$$

is continuous. We prove that $f(u) = 0$ for some $u \in \mathbb{C}$. Then also $p(u) = 0$ and u is a root of $p(z)$.

First we prove that f attains on its domain $\mathbb{C}$ a minimum value $f(u)$. Then we prove that $f(u) = 0$. Let the real number $K > 0$ be so large that

$$\tfrac{1}{2} \cdot K^n |a_n| > |a_0| \ \text{ and } \ \textstyle\sum_{j=0}^{n-1} |a_j| K^{j-n} < \tfrac{1}{2} \cdot |a_n| \, .$$

Then for every $z \in \mathbb{C}$ we have the estimate that

$$\begin{aligned} |z| > K \Rightarrow f(z) = |p(z)| \ &\geq \ |z|^n \big(|a_n| - \textstyle\sum_{j=0}^{n-1} |a_j| \cdot |z|^{j-n}\big) \\ &> \ |a_0| = |p(0)| = f(0) \ . \end{aligned}$$

We consider the rectangle

$$R = \{a + bi\colon \ -K \leq a,\, b \leq K\} \subset \mathbb{C} \ .$$

Clearly, if $z \in \mathbb{C} \setminus R$ then $|z| > K$. By Theorem 15 in the second lecture (the minimax principle) and Exercise 5 in this lecture there exists $u \in R$ such that $f(u) \leq f(v)$ for every $v \in R$. Since $0 \in R$, $f(u) \leq f(0)$. By the above estimate we have $f(u) \leq f(v)$ for every $v \in \mathbb{C}$. So f attains at u the smallest value on $\mathbb{C}$.

We prove that $f(u) = 0$. To this end we express the polynomial $p(z)$ by Exercise 7 as

$$p(z) = \textstyle\sum_{j=0}^{n} b_j (z - u)^j \, ,$$

with $b_j \in \mathbb{C}$ and $b_n = a_n$. So $f(u) = |p(u)| = |b_0|$. Let for contrary $f(u) = |b_0| > 0$. We write $p(z)$ as

$$p(z) = b_0 + b_k (z - u)^k + \underbrace{b_{k+1}(z - u)^{k+1} + \cdots + b_n (z - u)^n}_{q(z)} \, ,$$

where $q \in \mathbb{C}[z]$, $k \in \mathbb{N}$ and $b_0, b_k \neq 0$. We use the assumption of existence of roots in $\mathbb{C}$ and take an $\alpha \in \mathbb{C}$ such that

$$\alpha^k = -b_0/b_k \, .$$

It is clear that $q(z) = o((z-u)^k)$ (for $z \to u$), so that

$$\lim_{z \to u} q(z)(z-u)^{-k} = 0 \, .$$

Hence we can take a $\delta \in (0,1)$ such that for

$$v = u + \delta\alpha$$

one has

$$|q(v)| < \delta^k \cdot |b_0|/2 \, .$$

We get the contradiction that $f(v) < f(u)$:

$$\begin{aligned} f(v) = |p(v)| \quad & = & |b_0 + b_k \alpha^k \delta^k + q(v)| \\ & \overset{\text{def. of } \alpha}{=} & |b_0(1-\delta^k) + q(v)| \\ & \overset{\Delta\text{'s ineq. and mult. } |\cdot|}{\leq} & |b_0|(1-\delta^k) + |q(v)| \\ & \overset{|q(v)| < \ldots}{<} & |b_0|(1-\delta^k/2) \\ & \overset{\delta \in (0,1)}{<} & |b_0| = f(u) \, . \end{aligned}$$

Hence $f(u) = 0$ and $p(u) = 0$. □

Exercise 7 *Let $n \in \mathbb{N}_0$ and $u, a_0, a_1, \ldots, a_n \in \mathbb{C}$. Prove that then there exist complex numbers b_0, b_1, $\ldots$, b_n such that $b_n = a_n$ and*

$$\sum_{j=0}^n a_j z^j = \sum_{j=0}^n b_j (z-u)^j \, .$$

• *Complete metric spaces and complete sets.* A metric space (M, d) is *complete* if every Cauchy sequence $(a_n) \subset M$ is convergent. Recall that (a_n) is Cauchy if for every ε there is n_0 such that

$$m,\, n \geq n_0 \Rightarrow d(a_m,\, a_n) < \varepsilon\,.$$

A set $X \subset M$ is *complete* if the subspace (X, d) is complete.

Exercise 8 *Let (M, d) be metric space and $X \subset Y \subset M$. Prove that X is complete in the metric space (Y, d) if and only if X is complete in the metric space (M, d).*

Exercise 9 *Prove that the Cartesian product*

$$(M \times N,\, d \times e)$$

of complete metric spaces (M, d) and (N, e) is a complete MS.

A basic example of a complete metric space is the Euclidean space

$$(\mathbb{R},\, e_1) = (\mathbb{R},\, |x - y|)\,.$$

Its completeness is due to the fact that every sequence $(a_n) \subset \mathbb{R}$ is convergent if and only if it is Cauchy. By Exercise 9 all Euclidean spaces $(\mathbb{R}^n, e_n)$, $n \in \mathbb{N}$, are complete. We can construct many complete MSs as follows.

Proposition 10 *In every complete metric space (M, d) every closed subset $X \subset M$ is complete.*

Proof. Let $(a_n) \subset X$ be a Cauchy sequence in the closed set $X \subset M$. There exists $a = \lim a_n \in M$. Since X is a closed set, $a \in X$. So the set X is complete. □

Exercise 11 *Let $X \subset M$ be a compact set in a metric space (M, d). Prove that X is complete.*

Exercise 12 *Give an example of a complete and non-compact set $X \subset \mathbb{R}$ in the Euclidean metric space $(\mathbb{R}, e_1)$.*

Exercise 13 *Which of the following implications holds in a metric space (M, d)?*

1. *$X \subset M$ is a complete set $\Rightarrow$ X is closed.*
2. *$X \subset M$ and $Y \subset M$ are complete sets $\Rightarrow$ $X \cup Y$ is a complete set.*
3. *$X \subset M$ and $Y \subset M$ are complete sets $\Rightarrow$ $X \cap Y$ is a complete set.*
4. *$X \subset M$ is a complete set $\Rightarrow$ X is bounded.*
5. *$X \subset M$ is finite $\Rightarrow$ X is complete.*

• *Baire's theorem.* This is the main result about complete metric spaces. It says that no complete metric space is a countable union of sparse sets. A set $X \subset M$ in a metric space (M, d) is *sparse* if every ball $B(a, r) \subset M$ has a subball $B(b, s) \subset B(a, r)$ such that $B(b, s) \cap X = \emptyset$. A set $X \subset M$ is *dense* if for every ball $B(a, r) \subset M$ we have $B(a, r) \cap X \neq \emptyset$.

Exercise 14 *Let (M, d) be a metric space and $X \subset M$. Prove the equivalence*

$$X \text{ is dense} \iff \forall\, a \in M \;\exists\, (a_n) \subset X \colon \lim a_n = a\,.$$

Proposition 15 *Let (M, d) and (N, e) be metric spaces, $X \subset M$ be dense and let $f, g \colon M \to N$ be continuous maps such that $f \,|\, X = g \,|\, X$. Then $f = g$.*

Proof. Let $a \in M$ be arbitrary. By the previous exercise there exists a sequence $(a_n) \subset X$ such that $\lim a_n = a$. By Heine's definition of continuity and the assumption,

$$f(a) = \lim\, f(a_n) = \lim\, g(a_n) = g(a).$$

So $f = g$. □

Exercise 16 *Any finite union of sparse sets is a sparse set. This is in general not true for countable unions.*

Exercise 17 *The intersection of two dense sets, one of which is open, is a dense set. This is in general not true if we omit the assumption of openness.*

Let (M, d) be a metric space, $a \in M$ and let $r > 0$. The *closed ball* is

$$\overline{B}(a, r) = \{x \in M \colon\; d(a, x) \leq r\}\;.$$

Exercise 18 *Every closed ball is a closed set.*

Exercise 19 *A set $X \subset M$ in a metric space (M, d) is sparse if and only if every closed ball in M contains a closed subball disjoint to X.*

Theorem 20 (Baire's) *Let (M, d) be a complete metric space and let*

$$M = \bigcup\nolimits_{n=1}^{\infty} X_n\,.$$

Then some set X_n is not sparse. In other words, no complete metric space is a countable union of sparse sets.

Proof. We assume that all X_n are sparse and deduce a contradiction: we define a sequence $(\overline{B_n})$ of nested closed balls with centers converging to a point $a \in M$ that lies outside every X_n, which is a contradiction.

We use the definition of sparseness in Exercise 19. We take any $\overline{B}(b,1) \subset M$. Since X_1 is sparse, some $\overline{B}(a_1, r_1) \subset \overline{B}(b,1)$ is disjoint to X_1. Suppose that we already defined closed balls

$$\overline{B}(a_1,\, r_1) \supset \overline{B}(a_2,\, r_2) \supset \cdots \supset \overline{B}(a_n,\, r_n)$$

such that $\overline{B}(a_i, r_i) \cap X_i = \emptyset$ and $r_i \le 1/i$. Since X_{n+1} is sparse, some $\overline{B}(a_{n+1}, r_{n+1}) \subset \overline{B}(b_n, r_n)$ is disjoint to it. We may take the radius $r_{n+1} \le 1/(n+1)$. The sequence $(a_n) \subset M$ of centers is Cauchy:

$$m \ge n \Rightarrow \overline{B}(a_m,\, r_m) \subset \overline{B}(a_n,\, r_n) \text{ and } d(a_m,\, a_n) \le r_n \le 1/n\,.$$

Since (M, d) is complete, we have

$$a = \lim a_n \in M\ .$$

Since $m \ge n \Rightarrow a_m \in \overline{B}(a_n, r_n)$ and since by Exercise 18 every $\overline{B}(a_n, r_n)$ is a closed set, a lies in every closed ball $\overline{B}(a_n, r_n)$ and therefore in none of the sets X_n, which is a contradiction. $\square$

Baire's theorem has many applications, but now we mention only one. A point $a \in M$ in a metric space (M, d) is *isolated* if

$$\exists\, r > 0\colon B(a,\, r) = \{a\}\,.$$

Exercise 21 *Prove that in any metric space* (M,d),

$$a \in M \text{ is not isolated} \iff \{a\} \text{ is a sparse set.}$$

Corollary 22 *Any complete metric space without isolated points is uncountable.*

Proof. Suppose for the contrary that M is countable. Then

$$M = \bigcup_{a \in M} \{a\}$$

is a countable union. Since each set $\{a\}$ is sparse (by the previous exercise), we have a contradiction with Baire's theorem. $\square$

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 7, 11, 13, 16 and 21.