MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 3 (March 2, 2026) CONTINUITY AND COMPACTNESS. THE HEINE–BOREL THEOREM. CONNECTEDNESS

• *Compactness and continuity.* In the next exercise you verify that restriction of a continuous function to a subspace is a continuous function.

Exercise 1 *Let (M,d) and (N,e) be metric spaces, $X \subset M$ be a non-empty set and $f\colon M \to N$ be a continuous function. Then the restriction*

$$f\,|\,X\colon X \to N,\ \ X \ni a \mapsto f(a) \in N\ ,$$

defined on the subspace (X,d) is a continuous function.

Last time we considered two equivalent forms of continuity: (i) the classical ε-δ form and (ii) Heine's definition (based on limits of sequences). We introduce the third equivalent form of continuity, so called *topological continuity.*

Proposition 2 (topological continuity) *Let $f\colon M \to N$ be a map between metric spaces (M,d) and (N,e). Then f is continuous $\iff$ for every open set $A \subset N$, the set*

$$f^{-1}[A] = \{x \in M\colon\ f(x) \in A\}\ \ (\subset M)$$

is an open set in M.

Proof. The implication $\Rightarrow$. Let f be continuous in the ε-δ sense, $A \subset N$ be an open set and let $a \in f^{-1}[A]$. So $f(a) \in A$ and there exists ε such that $B(f(a), \varepsilon) \subset A$. Then there exists δ such that

$$f[B(a,\,\delta)] \subset B(f(a),\,\varepsilon) \subset A\,.$$

Hence $B(a, \delta) \subset f^{-1}[A]$ and $f^{-1}[A]$ is an open set.

The implication $\Leftarrow$. Let f be continuous in the topological sense, $a \in M$ and let an $\varepsilon > 0$ be given. Since the ball $B(f(a), \varepsilon)$ is an open set in N, $f^{-1}[B(f(a), \varepsilon)]$ is an open set in M. Since $a \in f^{-1}[B(f(a), \varepsilon)]$, there exists δ such that $B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]$. Thus

$$f[B(a,\,\delta)] \subset B(f(a),\,\varepsilon)$$

and f is continuous in the ε-δ sense. $\square$

Exercise 3 *Prove this equivalence with closed sets instead of open sets.*

We generalize the topological definition of continuity to subspaces.

Exercise 4 *Let (M, d) and (N, e) be metric spaces, $X \subset M$ and let $f\colon X \to N$. Then f is continuous as as map defined on the subspace (X, d) $\iff$ for every open set $A \subset N$ there is an open set $B \subset M$ such that*

$$f^{-1}[A] = X \cap B\,.$$

We show that the continuous image of a compact set is compact.

Proposition 5 *Suppose that (M, d) and (N, e) are metric spaces, $X \subset M$ is a compact set and $f\colon X \to N$ is a continuous function. Then the image $f[X] \subset N$ is a compact set.*

Proof. Let $(a_n) \subset f[X]$. We take the sequence $(b_n) \subset X$ with $f(b_n) = a_n$, and a convergent subsequence (b_{m_n}) with $\lim b_{m_n} = b \in X$. By Heine's definition of continuity,

$$\lim a_{m_n} = \lim f(b_{m_n}) = f(b) \in f[X]\,.$$

We have obtained a convergent subsequence in (a_n) with the limit in $f[X]$. So $f[X]$ is compact. □

Exercise 6 *Find an example showing that the inverse image of a compact set by a continuous function need not be compact.*

Another useful property of compact sets is the following result on continuity of inverses.

Proposition 7 *Let $f\colon X \to N$ be an injective continuous map from a compact set $X \subset M$ in a metric space (M, d) to a metric space (N, e). Then the inverse map*

$$f^{-1}\colon f[X] \to X$$

is continuous.

Proof. We use the topological continuity in Exercise 3. We need to prove that for every set $A \subset X$ that is closed in the subspace (X, d), the inverse image $(f^{-1})^{-1}[A] = f[A] \subset f[X]$ by the map f^{-1} is closed in the subspace $(f[X], e)$. By one of the exercises in the last lecture we know that A is compact because it is a closed set in a compact space. By the previous proposition, we know that $f[A]$ is a compact set in the subspace $(f[X], e)$. By a proposition in the last lecture, $f[A]$ is closed in this subspace. □

• *Homeomorphisms of metric spaces.* A map $f\colon M \to N$ between metric spaces (M, d) and (N, e) is their *homeomorphism* if f is a bijection and if both maps f and f^{-1} are continuous. If there is a homeomorphism between (M, d) and (N, e), these spaces are called *homeomorphic*.

Exercise 8 *Describe a homeomorphism between the Euclidean spaces* $(0, 1)$ *(*$\subset \mathbb{R}$*) and* $\mathbb{R}$.

Exercise 9 *Consider the Euclidean spaces* $I = [0, 2\pi)$ *(*$\subset \mathbb{R}$*) and*

$$S = \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \quad (\subset \mathbb{R}^2)\,.$$

Is the map $I \ni t \mapsto (\cos t, \sin t) \in S$ *a homeomorphism between them?*

Exercise 10 *Let* (M, d) *and* (N, e) *be homeomorphic metric spaces. Is it true that* M *is compact* $\iff$ N *is compact, and that* M *is bounded* $\iff$ N *is bounded?*

• *The Heine–Borel theorem.* This theorem characterizes compact sets in MSs by means of open sets. We say that a subset $A \subset M$ of a metric space (M, d) is *topologically compact* if for every system of open sets $\{X_i\colon\ i \in I\}$ in M with $\bigcup_{i \in I} X_i \supset A$ there exists a finite set $J \subset I$ such that $\bigcup_{i \in J} X_i \supset A$. One says that “every open cover of A has a finite subcover”. We prove that this is equivalent to the original definition of compactness.

Theorem 11 (Heine–Borel) *A set* $A \subset M$ *in a metric space* (M, d) *is compact* $\iff$ A *is topologically compact.*

Proof. Without loss of generality, $A = M$ (Exercise 12). We prove the implication $\Rightarrow$. Let (M, d) be a compact metric space and $M =$

$\bigcup_{i\in I} X_i$ be an open cover of it. We find in it a finite subcover. We first observe that for every δ there is a finite set $S_\delta \subset M$ such that

$$\bigcup_{a\in S_\delta} B(a,\, \delta) = M\,.$$

Suppose that this is not the case. It follows that then for some $\delta_0 > 0$ there is a sequence $(a_n) \subset M$ such that $m < n \Rightarrow d(a_m, a_n) \geq \delta_0$. But then (a_n) has no convergent subsequence, which contradicts the assumption.

Now we assume for the contrary that the above open cover of M by the sets X_i has no finite subcover. It follows, with the above finite sets S_δ, that **for every $n \in \mathbb{N}$ there is a point $\mathbf{b_n} \in \mathbf{S_{1/n}}$ such that**

$$\forall\, \mathbf{i} \in \mathbf{I}\colon \mathbf{B(b_n,\, 1/n)} \not\subset \mathbf{X_i}\,.$$

Indeed, else for some m every ball $B(b, 1/m)$, $b \in S_{1/m}$, would be contained in some X_i, and we would obtain a finite subcover, contrary to the assumption.

The boldface claim therefore holds and we consider the sequence $(b_n) \subset M$. By the assumption it has a convergent subsequence (b_{k_n}) with the limit

$$b = \lim_{n\to\infty} b_{k_n} \in M\,.$$

We take an index $j \in I$ such that $b \in X_j$, and a radius $r > 0$ for which $B(b, r) \subset X_j$. Let $n \in \mathbb{N}$ be so large that $1/k_n < r/2$ and $d(b, b_{k_n}) < r/2$. Then for every $x \in B(b_{k_n}, 1/k_n)$ we have $d(x, b) \leq d(x, b_{k_n}) + d(b_{k_n}, b) < r/2 + r/2 = r$. Hence

$$B(b_{k_n},\, 1/k_n) \subset B(b,\, r) \subset X_j\,,$$

which contradicts the boldface property of points b_n. Since the assumption that no finite subcover exists leads to a contradiction, the cover of M by the sets X_i, $i \in I$, has a finite subcover.

We prove the implication $\Leftarrow$. We assume that every open cover of M has a finite subcover, take a sequence $(a_n) \subset M$ and deduce that $(a_n) \subset M$ has a convergent subsequence.

We first assume that for every $b \in M$ there is a radius $r_b > 0$ such that the set

$$M_b = \{n \in \mathbb{N}\colon\ a_n \in B(b,\, r_b)\}$$

is finite, and show that this leads to a contradiction. Indeed, we would chose from the cover $M = \bigcup_{b \in M} B(b, r_b)$ a finite subcover and, since a finite union of finite sets is finite, we would deduce that for almost all n the term a_n lies outside this subcover, that is, outside M.

So it holds on the contrary that **there is a point $\mathbf{b} \in \mathbf{M}$ such that for every $\mathbf{r} > \mathbf{0}$ the set**

$$\mathbf{M_r} = \{\mathbf{n} \in \mathbf{N}\colon\ \mathbf{a_n} \in \mathbf{B}(\mathbf{b},\, \mathbf{r})\}$$

is infinite. Now we easily select from (a_n) a convergent subsequence (a_{k_n}) with the limit b. Let the indices $1 \le k_1 < k_2 < \cdots < k_n$ be already defined such that $d(b, a_{k_i}) < 1/i$ for $i = 1, 2, \ldots, n$. The set of indices $M_{1/(n+1)}$ is infinite, so we can choose a $k_{n+1} \in \mathbb{N}$ such that $k_{n+1} > k_n$ and $k_{n+1} \in M_{1/(n+1)}$. Then also $d(b, a_{k_{n+1}}) < 1/(n+1)$. We get a subsequence (a_{k_n}) of (a_n) converging to b. $\square$

Exercise 12 *Why can one take in the previous proof $A = M$?*

• *Connected sets in metric spaces.* The subset $X \subset M$ in a metric space (M, d) is *clopen* if it is both open and closed. For example, the sets $\emptyset$ and M are clopen. The space M is *connected* if it has no nontrivial—different from $\emptyset$ and M—clopen subset. Else, if M has a clopen subset $X \subset M$ different from $\emptyset$ and M, we say that M is *disconnected.* A subset $X \subset M$ is *connected*, or *disconnected*, if the subspace (X, d) *is connected*, or *disconnected.*

Exercise 13 *Which finite sets $X \subset \mathbb{R}$ in the Euclidean space $\mathbb{R}$ are connected?*

Exercise 14 *Is the set*

$$X = (\{0\} \times [-1,\, 1]) \cup \{(t,\, \sin(1/t))\colon\; 0 < t \le 1\}$$

in the Euclidean plane $\mathbb{R}^2$ connected?

Let (M, d) be a metric space and $X, A, B \subset M$. We say that the sets A and B *cut the set* X if A and B are open and

$$(X \subset A \cup B) \wedge (X \cap A \neq \emptyset \neq X \cap B) \wedge (X \cap A \cap B = \emptyset)\;.$$

Exercise 15 *Prove that $X \subset M$ is a disconnected set in a metric space (M, d) if and only if there exist sets $A, B \subset M$ that cut X.*

Exercise 16 *Let (M, d) be a metric space and $A, B \subset M$ be connected sets such that $A \cap B \neq \emptyset$. Prove that $A \cup B$ is connected.*

• *The Fundamental Theorem of Algebra.* We prove it using compact and continuous sets in the metric space $\mathbb{C}$.

Theorem 17 (FTA) *Every non-constant complex polynomial has a root.*

We prepare for the proof by deriving a couple of results on connectedness. Regarding compactness, we are ready: the metric space

$\mathbb{C} = (\mathbb{C}, |u - v|)$ is actually the Euclidean space $(\mathbb{R}^2, e_2)$, and a set $X \subset \mathbb{C}$ is compact iff X is closed and bounded.

We view the real axis $\mathbb{R}$ as contained in $\mathbb{C}$. We prove that every interval $[a, b] \subset \mathbb{R}$ $(\subset \mathbb{C})$ is a connected set in $\mathbb{C}$.

Theorem 18 (intervals are connected) *Let a, b be in $\mathbb{R}$ and $a \le b$. Then the interval $[a, b]$ is a connected set.*

Proof. We use Exercise 15 and suppose, for the conrary, that $A, B \subset \mathbb{C}$ are open sets that cut $[a, b]$. We can assume that $a < b$ and that $a \in A$ and $b \in B$ (Exercise 19). We consider the number

$$c = \sup(\{x \in [a,\, b] \mid x \in A\}) \in [a,\, b]\ .$$

Then $c \in A \cup B$. If $c \in A$, then $c < b$. Since A is open, every c' with $c < c' < b$ and sufficiently close to c lies in A. But this contradicts that c is an upper bound of the set $A \cap [a, b]$. If $c \in B$ then $a < c$. Now we see similarly that every c' with $a < c' < c$ and sufficiently close to c lies in B, that is, outside of A. But this contradicts the fact that c is the smallest upper bound of the set $A \cap [a, b]$. $\square$

Exercise 19 *Why can we assume that $a \in A$ and $b \in B$?*

Exercise 20 *Prove the equivalence*

$$X \subset \mathbb{R} \textit{ is connected} \iff X \textit{ is an interval}\ .$$

Like compact sets, connected sets are preserved by continuous images.

Theorem 21 (continuity and connectedness) *We assume that $f\colon X \to N$ is a continuous map from a connected set*

$X \subset M$ in a metric space (M, d) to another metric space (N, e). Then the image

$$f[X] = \{f(x)\colon\ x \in X\} \quad (\subset N)$$

is connected.

Proof. We prove that if $f[X]$ is disconnected, then so is X. Let some open sets $A, B \subset N$ cut $f[X]$. By Exercise 4 there exist open sets $A', B' \subset M$ such that

$$f^{-1}[A] = X \cap A' \ \text{ and } \ f^{-1}[B] = X \cap B'\ .$$

It follows that A' and B' cut X. So the set X is disconnected. □

We prove that the *complex unit circle*

$$S = \{z \in \mathbb{C} \mid |z| = 1\} \quad (\subset \mathbb{C})$$

is connected. We could take the continuous function $f(t) = \cos t + i \sin t\colon\ I = [0, 2\pi] \to \mathbb{C}$. Then

$$S = f[I]\,,$$

and S is connected by the two previous theorems. If we want to avoid the transcendental functions sin and cos, we can use the algebraic functions $f^+, f^-\colon I = [-1, 1] \to \mathbb{C}$, defined by

$$f^+(t) = t + i\sqrt{1 - t^2} \ \text{ and } \ f^-(t) = t - i\sqrt{1 - t^2}\ .$$

Then

$$S = f^+[I] \cup f^-[I]\,,$$

and S is connected by the two previous theorems and Exercise 16.

We turn to the first of two steps in the proof of FTA and prove that $\mathbb{C}$ contains every n-th root. We again avoid sine and cosine. Two special cases of this result are left as exercises.

Exercise 22 *Prove that for every real $x \geq 0$ and every $n \in \mathbb{N}$ there exists a real $y \geq 0$ such that $y^n = x$.*

Exercise 23 (square roots in $\mathbb{C}$) *Let $a + bi \in \mathbb{C}$. Then, for an appropriate choice of signs in the formulas*

$$c = \pm\frac{\sqrt{\sqrt{a^2+b^2}+a}}{\sqrt{2}} \ \text{ and } \ d = \pm\frac{\sqrt{\sqrt{a^2+b^2}-a}}{\sqrt{2}}\,,$$

we have $(c+di)^2 = a+bi$. What are these signs? How would you derive these formulas? (Checking their correctness is easy.)

Theorem 24 (nth roots in $\mathbb{C}$) *For every number $u \in \mathbb{C}$ and every exponent $n \in \mathbb{N}$ there exists a number $v \in \mathbb{C}$ such that*

$$v^n = u\,.$$

We prove it in the next lecture. Then we also complete the proof of FTA by the second step based on compact sets.

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 4, 9, 13, 14 and 23.