MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
lecturer: Martin Klazar

LECTURE 3 (March 5, 2025) CONTINUITY AND COMPACTNESS. THE HEINE–BOREL THEOREM. CONNECTEDNESS. FTALG

• *Compactness and continuity.* In the next exercise you verify that restriction of a continuous function to a subspace is a continuous function.

Exercise 1 *Let* (M,d) *and* (N,e) *be* MSs*,* $X \subset M$ *be a non-empty set and* $f\colon M \to N$ *be a continuous function. Then the restriction*

$$f \mid X \colon X \to N,\ X \ni a \mapsto f(a) \in N \ ,$$

defined on the subspace (X,d) *is a continuous function.*

In the last lecture, we met two equivalent versions of continuity of functions: (i) the classical ε-δ form and (ii) the Heine definition (based on limits of sequences). Now we introduce the third equivalent form of continuity, so called *topological continuity.*

Proposition 2 (topological continuity) *Let* $f\colon M \to N$ *be a map between* MSs (M,d) *and* (N,e)*. Then, with* OS *standing for "open set",*

$$\begin{aligned}&f \text{ is continuous} \iff\\ &\forall\,\mathrm{OS}\ A \subset N\ \big(f^{-1}[A] = \{x \in M \mid f(x) \in A\} \subset M \text{ is an } \mathrm{OS}\big)\ .\end{aligned}$$

Proof. The implication $\Rightarrow$. Let f be continuous in the ε-δ sense, $A \subset N$ be an open set and $a \in f^{-1}[A]$. So $f(a) \in A$ and there exists an $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subset A$. So there exists a $\delta > 0$ that

$$f[B(a,\,\delta)] \subset B(f(a),\,\varepsilon) \subset A\,.$$

Hence $B(a, \delta) \subset f^{-1}[A]$ and $f^{-1}[A]$ is an open set.

The implication $\Leftarrow$. Let f be continuous in the topological sense, $a \in M$ and $\varepsilon > 0$. Since the ball $B(f(a), \varepsilon) \subset N$ is an open set, $f^{-1}[B(f(a), \varepsilon)]$ is an open set. Since $a \in f^{-1}[B(f(a), \varepsilon)]$, there exists a $\delta > 0$ such that $B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]$. Thus

$$f[B(a,\,\delta)] \subset B(f(a),\,\varepsilon)$$

and f is continuous in the ε-δ sense. □

Exercise 3 *Prove this equivalence with closed sets instead of open sets.*

We generalize the topological definition of continuity to subspaces.

Exercise 4 *Let* (M, d) *and* (N, e) *be* MS*s,* $X \subset M$ *and let* $f\colon X \to N$*. Then (*OS *is again an "open set")*

$$f \text{ is a continuous map defined on the subspace } (X, d) \iff \\ \iff \forall\,\mathrm{OS}\ A \subset N\ \exists\,\mathrm{OS}\ B \subset M\ \big(f^{-1}[A] = X \cap B\big)\,.$$

We show that the continuous image of a compact set is compact.

Proposition 5 (compact image) *Let* (M, d) *and* (N, e) *be* MS*s,* $X \subset M$ *be a compact set and*

$$f\colon X \to N$$

be a continuous function. Then the image $f[X] \subset N$ is a compact set.

Proof. Let $(a_n) \subset f[X]$ be an arbitrary sequence. We take the sequence $(b_n) \subset X$ with $f(b_n) = a_n$ and select a convergent subsequence (b_{m_n}) with $\lim b_{m_n} = b \in X$. By Heine's definition of continuity,

$$\lim a_{m_n} = \lim f(b_{m_n}) = f(b) \in f[X] \ .$$

We have obtained a convergent subsequence of the sequence (a_n) with limit in $f[X]$. So $f[X]$ is compact. $\square$

Exercise 6 *Find an example showing that the inverse image of a compact set by a continuous function need not be compact.*

Another useful property of compact sets is the following.

Proposition 7 (continuity of inverses) *Let*

$$f \colon X \to N$$

be an injective continuous map from a compact set $X \subset M$ in a MS (M, d) *to a* MS (N, e)*. Then the inverse map*

$$f^{-1} \colon f[X] \to X$$

is continuous.

Proof. We use the version of topological continuity in Exercise 3. We need to prove that for every set $A \subset X$ that is closed in the subspace (X, d), the inverse image $(f^{-1})^{-1}[A] = f[A] \subset f[X]$ by the map f^{-1} is closed in the subspace $(f[X], e)$. By one of the exercises in the last lecture we know that A is compact (it is a closed set in a compact space). By the previous proposition, we know that $f[A]$ is a compact set in the subspace $(f[X], e)$. By a proposition in the last lecture, $f[A]$ is closed in this subspace. $\square$

• *Homeomorphisms of* MS*s.* A map $f\colon M \to N$ between MSs (M, d) and (N, e) is their *homeomorphism* if f is a bijection and if both f and f^{-1} are continuous. If there is a homeomorphism between (M, d) and (N, e), these spaces are called *homeomorphic*.

Exercise 8 *Describe the homeomorphism between the Euclidean spaces* $(0, 1) \subset \mathbb{R}$ *and* $\mathbb{R}$*.*

Exercise 9 *Consider the Euclidean spaces* $I = [0, 2\pi) \subset \mathbb{R}$ *and the unit circle*

$$S = \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \subset \mathbb{R}^2\ .$$

Is the mapping $I \ni t \mapsto (\cos t, \sin t) \in S$ *a homeomorphism between them?*

Exercise 10 *Let* (M, d) *and* (N, e) *be homeomorphic* MP*s. Is it true that* M *is compact* $\iff$ N *is compact, and that* M *is bounded* $\iff$ N *is bounded?*

• *The Heine–Borel theorem.* This theorem characterizes compact sets in MSs by means of open sets. We say that a subset $A \subset M$ of a MS (M, d) is *topologically compact* if for every system of open sets $\{X_i \mid i \in I\}$ in M it holds that

$$\bigcup_{i\in I} X_i \supset A \Rightarrow \exists \text{ finite set } J \subset I \left(\bigcup_{i\in J} X_i \subset A\right).$$

One says that "every open covering of A has a finite subcovering". We prove that this definition of compactness is equivalent to the original definition.

Theorem 11 (Heine–Borel) *A set $A \subset M$ in a metric space (M, d) is compact if and only if it is topologically compact.*

Proof. Without loss of generality, $A = M$ (Exercise 12).

We prove the implication $\Rightarrow$. Let (M, d) be a compact MS and

$$M = \bigcup_{i \in I} X_i$$

be its open covering (so every set X_i is open). We find a finite subcovering in the system

$$\{X_i \mid i \in I\} \ .$$

First we prove that

$$\forall\, \delta > 0 \ \exists \text{ finite set } S_\delta \subset M \left(\bigcup_{a \in S_\delta} B(a,\, \delta) = M \right) .$$

If this were not the case, there would exist a $\delta_0 > 0$ and a sequence $(a_n) \subset M$ such that $m < n \Rightarrow d(a_m, a_n) \geq \delta_0$. In contrary with the assumed compactness of the set M this sequence has no convergent subsequence. Indeed, if (we negate the above statement about δ and S_δ) there exists a $\delta_0 > 0$ such that for every finite set $S \subset M$ one has that

$$M \setminus \textstyle\bigcup_{a \in S} B(a,\, \delta_0) \neq \emptyset \ ,$$

then — if we already have defined points $a_1,\, a_2,\, \ldots,\, a_n$ satisfying that $d(a_i, a_j) \geq \delta_0$ for every $1 \leq i < j \leq n$ — we take $a_{n+1} \in$

$M \setminus \bigcup_{i=1}^{n} B(a_i, \delta_0)$ and a_{n+1} has from each point a_1, a_2, …, a_n distance at least δ_0. Thus we define the whole sequence (a_n).

For contrary we assume that the above open covering of M by the sets X_i has no finite subcovering. We argue that it follows that (the finite sets S_δ are defined above)

$$\forall\, n \in \mathbb{N}\, \exists\, b_n \in S_{1/n}\, \forall\, i \in I\, \big(B(b_n, 1/n) \not\subset X_i\big)\ .$$

If this were not the case, then (negating the previous statement) there would exist an $n_0 \in \mathbb{N}$ such that for every $b \in S_{1/n_0}$ there exists a $i_b \in I$ such that $B(b, 1/n_0) \subset X_{i_b}$. But then, since $M = \bigcup_{b \in S_{1/n_0}} B(b, 1/n_0)$, the indices give $J = \{i_b \mid b \in S_{1/n_0}\} \subset I$ in contrary with the assumption on finite subcovering of the set M.

The displayed claim on n and b_n is therefore valid and we have the sequence $(b_n) \subset M$. By the assumption it has a convergent subsequence (b_{k_n}) with $b = \lim b_{k_n} \in M$. Since the X_i cover M, there exists a $j \in I$ such that $b \in X_j$. Due to the openness of X_j there exists an $r > 0$ such that $B(b, r) \subset X_j$. We take $n \in \mathbb{N}$ so large that $1/k_n < r/2$ and $d(b, b_{k_n}) < r/2$. For every $x \in B(b_{k_n}, 1/k_n)$ then, by the triangle inequality, we have that $d(x, b) \le d(x, b_{k_n}) + d(b_{k_n}, b) < r/2 + r/2 = r$. Hence

$$B(b_{k_n},\, 1/k_n) \subset B(b,\, r) \subset X_j\ ,$$

in contrary with the above property of points b_n. The assumption that finite subcovering does not exist leads to a contradiction. Hence the cover of M by the sets X_i, $i \in I$, has a finite subcover.

We prove the implication $\Leftarrow$, which is easier. We assume that every open covering of the set M has a finite subcovering, and we derive from this that that every sequence $(a_n) \subset M$ has a convergent subsequence. We first assume that

$$\forall\, b \in M\ \exists\, r_b > 0 \left(M_b = \{n \in \mathbb{N} \mid a_n \in B(b,\, r_b)\} \text{ is finite}\right)$$

and show that this assumption leads to a contradiction. Indeed, from the covering $M = \bigcup_{b \in M} B(b, r_b)$ we would choose a finite subcovering given by a finite set $N \subset M$ and we would deduce that there exists an n_0 such that $n \geq n_0 \Rightarrow a_n \notin \bigcup_{b \in N} B(b, r_b)$ because the set of indices $\bigcup_{b \in N} M_b$ is finite (it is a finite union of finite sets). But this is a contradiction because $\bigcup_{b \in N} B(b, r_b) = M$. So the assumption does not hold and on the contrary it is true that

$$\exists\, b \in M\ \forall\, r > 0 \left(M_r = \{n \in N \mid a_n \in B(b, r)\}\ \text{ is infinite}\right)\ .$$

Now we can easily select from (a_n) a convergent subsequence (a_{k_n}) with the limit b. Let the indices $1 \leq k_1 < k_2 < \cdots < k_n$ be already defined such that $d(b, a_{k_i}) < 1/i$ for $i = 1, 2, \ldots, n$. The set of indices $M_{1/(n+1)}$ is infinite, so we can choose a $k_{n+1} \in \mathbb{N}$ such that $k_{n+1} > k_n$ and $k_{n+1} \in M_{1/(n+1)}$. Then also $d(b, a_{k_{n+1}}) < 1/(n+1)$. We get a subsequence (a_{k_n}) of (a_n) converging to b. □

Exercise 12 *Why can one take in the previous proof $A = M$?*

• *Connected sets and* MS*s.* The subset $X \subset M$ in a MS (M, d) is *clopen* if it is at the same time open and closed. For example, the sets $\emptyset$ and M clopen. The space M is *connected* if it has no nontrivial (different from $\emptyset$ and M) clopen subset. Else, if M has a clopen subset $X \subset M$ with $X \neq \emptyset, M$, we say that M is *disconnected*. A subset $X \subset M$ is *connected*, or *disconnected*, if the subspace (X, d) *is connected*, or *disconnected*.

Exercise 13 *Which finite sets $X \subset \mathbb{R}$ in the Euclidean space $\mathbb{R}$ are connected?*

Exercise 14 *Is the set* $X \subset \mathbb{R}^2$ *in the Euclidean plane* $\mathbb{R}^2$*, given as*

$$X = (\{0\} \times [-1,\, 1]) \cup \{(t,\, \sin(1/t)) \mid 0 < t \le 1\}\ ,$$

connected?

Let (M, d) be a MS and $X, A, B \subset M$. We say that the sets A and B *cut the set* X if A and B are open and

$$(X \subset A \cup B) \wedge (X \cap A \ne \emptyset \ne X \cap B) \wedge (X \cap A \cap B = \emptyset)\ .$$

Exercise 15 *Prove that* $X \subset M$ *is a disconnected set in a* MS (M, d) *if and only if there are sets* $A, B \subset M$ *that cut* X*.*

Exercise 16 *Let* (M, d) *be a* MP *and* $A, B \subset M$ *be connected sets such that* $A \cap B \ne \emptyset$*. Prove that then the set* $A \cup B$ *is connected.*

• *The Fundamental Theorem of Algebra* (FTAlg). We prove it using compact and continuous sets in the MS $\mathbb{C}$.

Theorem 17 (FTAlg) *Every non-constant complex polynomial has a root, that is,*

$$\begin{aligned}&(n \ge 1) \wedge (a_0,\, a_1,\, \ldots,\, a_n \in \mathbb{C}) \wedge (a_n \ne 0) \Rightarrow\\ &\Rightarrow \exists\, \alpha \in \mathbb{C}\, \left(\textstyle\sum_{j=0}^{n} a_j \alpha^j = 0\right)\ .\end{aligned}$$

However, we still have to derive some results on connected sets. From the point of view of compact sets, we are ready: the MS $\mathbb{C} = (\mathbb{C}, |u-v|)$ is actually the Euclidean space $(\mathbb{R}^2, e_2)$ and $X \subset \mathbb{C}$ is compact iff X is closed and bounded.

We regard the real axis $\mathbb{R}$ as contained in $\mathbb{C}$ and first we prove that every interval $[a, b] \subset \mathbb{R} \subset \mathbb{C}$ is a connected set in $\mathbb{C}$.

Theorem 18 (intervals are connected) *Let $a, b \in \mathbb{R}$ with $a \le b$. Then the interval $[a, b]$ ($\subset \mathbb{C}$) is a connected set.*

Proof. For contrary let $A, B \subset \mathbb{C}$ be open sets cutting $[a, b]$ (Exercise 15). We can assume that $a < b$ and that $a \in A$ and $b \in B$ (Exercise 19). We consider the number

$$c = \sup(\{x \in [a,\, b] \mid x \in A\}) \in [a,\, b]\ .$$

Then $c \in A \cup B$. If $c \in A$, then $c < b$. It follows from the openness of A that every c' with $c < c' < b$ and sufficiently close to c lies in A. But this contradicts that c is an upper bound of the set $A \cap [a, b]$. If $c \in B$, then $a < c$. It follows from the openness of B that every c' with $a < c' < c$ and sufficiently close to c lies in B, that is, outside of A. But this contradicts the fact that c is the smallest upper bound of the set $A \cap [a, b]$. □

Exercise 19 *Why can we assume that $a \in A$ and $b \in B$?*

Exercise 20 *Prove the equivalence*

$$X \subset \mathbb{R} \textit{ is connected} \iff X \textit{ is an interval}\,.$$

Connectedness is preserved by continuous functions (like compactness).

Theorem 21 (continuity and connectedness) *We assume that $f\colon X \to N$ is a continuous map from a connected set $X \subset M$ in a* MS (M, d) *to another* MS (N, e). *Then the image*

$$f[X] = \{f(x) \mid x \in X\} \subset N$$

is connected.

Proof. We deduce from the disconnectedness of $f[X]$ the disconnectedness of X. Let the open sets $A, B \subset N$ cut the set $f[X]$. By Exercise 4 there exist open sets $A', B' \subset M$ such that

$$f^{-1}[A] = X \cap A' \ \text{ and } \ f^{-1}[B] = X \cap B' \ .$$

It is easy to see that the sets A' and B' cut the set X. It is therefore disconnected. $\square$

Now we easily prove that the *complex unit circle*

$$S = \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C}$$

is connected. We could take the continuous function $f(t) = \cos t + i \sin t \colon I = [0, 2\pi] \to \mathbb{C}$. Then

$$S = f[I]\,,$$

and S is connected by the two previous theorems. However, we use the transcendental functions sin and cos. We can avoid them by replacing f with continuous functions $f^+, f^- \colon I = [-1, 1] \to \mathbb{C}$, defined by

$$f^+(t) = t + i\sqrt{1 - t^2} \ \text{ and } \ f^-(t) = t - i\sqrt{1 - t^2} \ .$$

Then

$$S = f^+[I] \cup f^-[I]\,,$$

and S is connected by the two previous theorems and Exercise 16.

We now turn to the first of two steps in the proof of FTAlg. We prove that $\mathbb{C}$ contains all n-th roots for $n \in \mathbb{N}$. We again avoid sine and cosine. Two special cases of this fact are left as exercises.

Exercise 22 *Prove that for every nonnegative $x \in \mathbb{R}$ and every $n \in \mathbb{N}$ there exists a nonnegative $y \in \mathbb{R}$ such that $y^n = x$.*

Exercise 23 (square roots in $\mathbb{C}$) *Let $a+bi \in \mathbb{C}$. Then, for appropriate choice of signs in*

$$c = \pm\frac{\sqrt{\sqrt{a^2+b^2}+a}}{\sqrt{2}} \;\; \text{and} \;\; d = \pm\frac{\sqrt{\sqrt{a^2+b^2}-a}}{\sqrt{2}}\,,$$

it holds that $(c+di)^2 = a+bi$. What are these signs? How would you derive these formulas? (Checking their correctness is easy.)

Theorem 24 (nth roots in $\mathbb{C}$) $\mathbb{C}$ *contains all n-th roots,*

$$\forall\, u \in \mathbb{C}\; \forall\, n \in \mathbb{N}\; \exists\, v \in \mathbb{C}\, \big(v^n = u\big)\;.$$

We prove it next time. Then we also complete the proof of FTAlg in the second step using compact sets.

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 4, 9, 13, 14 and 23.