MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 2 (February 23, 2026) OSTROWSKI'S THEOREM. COMPACT METRIC SPACES.

• *Ostrowski's theorem.* On any field F we have the *trivial norm*. It is a function $\| \cdot \|$ with $\|0_F\| = 0$ and $\|x\| = 1$ for $x \neq 0_F$.

Exercise 1 *Prove that a trivial norm is a norm.*

From the usual absolute value $| \cdot |$ on $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, we get many other norms by exponentiation.

Exercise 2 *Prove that for any $c > 0$, $| \cdot |^c$ is a norm (on $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) if and only if $c \leq 1$. We will call this norm the modified absolute value.*

For $\alpha \in \mathbb{Q}$ and a prime p, the *canonical p-adic norm* $\| \cdot \|_p$ is defined by
$$\|\alpha\|_p = p^{-\mathrm{ord}_p(\alpha)}$$
$-$ in the general p-adic norm $| \cdot |_p$ we set $c = 1/p$.

Exercise 3 *Let $M = \{2, 3, 5, 7, 11, \dots\} \cup \{\infty\}$ and $\| \cdot \|_\infty = | \cdot |$ (ordinary absolute value). Prove that for every nonzero number $\alpha \in \mathbb{Q}$ we have the product formula*
$$\prod_{p \in M} \|\alpha\|_p = 1 \, .$$

Exercise 4 *Let $\| \cdot \|$ be a nontrivial norm on the field $\mathbb{Q}$. Prove that $\exists \, n \in \mathbb{N} \, \big(n \geq 2 \wedge \|n\| \neq 1 \big)$.*

Exercise 5 *Prove that for every two coprime numbers $a, b \in \mathbb{Z}$ there exist numbers $c, d \in \mathbb{Z}$ such that*

$$ac + db = 1 .$$

Theorem 6 (A. Ostrowski, 1916) *Let $\|\cdot\|$ be a norm on the field $\mathbb{Q}$. Then (exactly) one of three cases occurs.*

1. *$\| \cdot \|$ is a trivial norm.*

2. *There exists a number $c \in (0, 1]$ such that $\|x\| = |x|^c$.*

3. *There exists a number $c \in (0, 1)$ and a prime number p such that $\|x\| = |x|_p = c^{\mathrm{ord}_p(x)}$.*

Thus modified absolute values and p-adic norms are the only non-trivial norms on the field of fractions.

Proof. Let $\| \cdot \|$ be a nontrivial norm. By Exercise 4 there exists an $n \in \mathbb{N} \setminus \{1\}$ such that $\|n\| \neq 1$. Two cases occur.

1. There is an $n \in \mathbb{N}$ such that $\|n\| > 1$. Let n_0 be the smallest such n. Clearly, $n_0 \geq 2$ a

$$1 \leq m < n_0 \Rightarrow \|m\| \leq 1 . \tag{1}$$

We define a (unique) real number $c > 0$ by

$$\|n_0\| = n_0^c . \tag{2}$$

Every number $n \in \mathbb{N}$ has a (unique) base n_0 expansion:

$$n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s \text{ where}$$
$$a_i, \ s \in \mathbb{N}_0, \ 0 \leq a_i < n_0 \ \text{ and } \ a_s \neq 0 .$$

For $n_0 = 10$ this is the usual decadic notation: for example, $2024 = 2 \cdot 10^3 + 0 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0$. We have

$$\|n\| \qquad = \qquad \|a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s\|$$

$$\overset{\Delta\text{-ineq. and multipl. of } \|\cdot\|}{\leq} \qquad \sum_{j=0}^{s} \|a_j\| \cdot \|n_0\|^j$$

$$\overset{\text{eq. (1) and (2)}}{\leq} \qquad \sum_{j=0}^{s} n_0^{jc} \leq n_0^{sc} \sum_{i=0}^{\infty} \left(1/n_0^c\right)^i$$

$$\overset{n_0^s \leq n}{\leq} \qquad n^c C \ \text{ where } \ C = \sum_{i=0}^{\infty} \left(1/n_0^c\right)^i .$$

Hence

$$\forall n \in \mathbb{N}_0 : \ \|n\| \leq C n^c . \tag{3}$$

This in fact holds with $C = 1$. Let $m, n \in \mathbb{N}$. By the multiplicativity of the norm and inequality (3),

$$\|n\|^m = \|n^m\| \leq C \left(n^m\right)^c = C \left(n^c\right)^m .$$

We take the m-th root and get $\|n\| \leq C^{1/m} n^c$. For $m \to \infty$ we have $C^{1/m} \to 1$. Thus

$$\forall n \in \mathbb{N}_0 \left(\|n\| \leq n^c\right) . \tag{4}$$

We derive the reverse inequality $\|n\| \geq n^c$, $n \in \mathbb{N}_0$. For every $n \in \mathbb{N}$, the above base n_0 expansion of n yields

$$n_0^{s+1} > n \geq n_0^s .$$

By the Δ-inequality,

$$\|n_0\|^{s+1} = \|n_0^{s+1}\| \leq \|n\| + \|n_0^{s+1} - n\| .$$

Hence

$$\|n\| \ \geq \ \|n_0\|^{s+1} - \|n_0^{s+1} - n\| \overset{(2) \text{ and } (4)}{\geq} n_0^{(s+1)c} - (n_0^{s+1} - n)^c$$

$$\overset{n \geq n_0^s}{\geq} \ n_0^{(s+1)c} - (n_0^{s+1} - n_0^s)^c = n_0^{(s+1)c}\left(1 - \left(1 - \tfrac{1}{n_0}\right)^c\right)$$

$$\overset{n_0^{s+1} > n}{\geq} \ n^c C' \ \text{ where } \ C' = 1 - \left(1 - \tfrac{1}{n_0}\right)^c > 0 \ .$$

The trick with the m-th root gives again

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \geq n^c\right) \ .$$

Hence

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| = n^c\right) \ .$$

From the multiplicativity of the norm $\|\cdot\|$, we get that $\|x\| = |x|^c$ for any $x \in \mathbb{Q}$. By Exercise 2, $c \in (0, 1]$. We get case 2 of Ostrowski's theorem.

2. $\underline{\forall n \in \mathbb{N} : \ \|n\| \leq 1 \text{ and } \exists n \in \mathbb{N} : \ \|n\| < 1}$. Let n_0 be minimum such n; again $n_0 \geq 2$. We claim that $n_0 = p$ is a prime number. Indeed, if we could express n_0 as $n_0 = n_1 n_2$ with $n_i \in \mathbb{Z}$ and $1 < n_1, n_2 < n_0$, we get the contradiction

$$1 > \|n_0\| = \|n_1 n_2\| = \|n_1\| \cdot \|n_2\| = 1 \cdot 1 = 1$$

—we used the multiplicativity of norms and that $\|m\| = 1$ for any $m \in \mathbb{N}$ with $1 \leq m < n_0$. We show that every prime number q different from p has the norm $\|q\| = 1$. For the contrary, let $q \neq p$ be another prime number with $\|q\| < 1$. We take a large $m \in \mathbb{N}$ such that $\|p\|^m, \|q\|^m < \tfrac{1}{2}$. By Exercise 5 there exist integers a and

b such that $aq^m + bp^m = 1$. Then

$$
\begin{aligned}
1 &= \|1\| = \|aq^m + bp^m\| \le \|a\| \cdot \|q\|^m + \|b\| \cdot \|p\|^m \\
&< 1 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{2} = 1
\end{aligned}
$$

which is a contradiction; we used the triangle inequality, multiplicativity of norms, and the fact that now $\|a\| \le 1$ for every $a \in \mathbb{Z}$.

Thus $\|q\| = 1$ for every prime number $q \ne p$. From this, multiplicativity of norms and prime factorization of any non-zero fraction x we get the expression

$$
\begin{aligned}
\|x\| &= \left\| \prod_{q=2,\,3,\,5,\,\dots} q^{\mathrm{ord}_q(x)} \right\| = \prod_{q=2,\,3,\,5,\,\dots} \|q\|^{\mathrm{ord}_q(x)} = \|p\|^{\mathrm{ord}_p(x)} \\
&= c^{\mathrm{ord}_p(x)}, \ \text{where } c = \|p\| \in (0,1) \,.
\end{aligned}
$$

Also $\|0\| = c^{\mathrm{ord}_p(0)} = c^\infty = 0$. Thus we get case 3 of Ostrowski's theorem. $\qquad\square$

The preceding proof is taken from the book by Neal Koblitz,

p-adic Numbers, p-adic Analysis, and Zeta-Functions, Springer-Verlag, New York, 1984.

This book contains lot of information about the p-adic norm $\| \cdot \|_p$ and related p-adic analysis.

• *Compact sets in metric spaces.* We review limits of sequences in metric spaces (MS). Let (M, d) be a MS, $(a_n) \subset M$ be a sequence of points in it and $a \in M$ be a point. We say that (a_n) has the *limit a* in the MS (M, d) if for every ε there is an n_0 such that

$$
n \ge n_0 \Rightarrow d(a_n,\, a) < \varepsilon \,.
$$

From now on $\varepsilon > 0$ is a real number and $n_0, n \in \mathbb{N}$. We write that $\lim a_n = a$ or $\lim_{n\to\infty} a_n = a$. If the sequence (a_n) has a limit, we say that it is *convergent*, otherwise it is *divergent*.

Let (M, d) be a MS and $X \subset M$. We say that the set X is *compact* if every sequence $(a_n) \subset X$ has a subsequence (a_{m_n}) such that

$$\lim_{n\to\infty} a_{m_n} = a \in X\,.$$

The MS (M, d) *is compact* when the set M is compact.

The Bolzano–Weierstrass theorem states that in the MS

$$(\mathbb{R},\ |x - y|)$$

every closed and bounded interval $X = [a, b]$ is a compact set. We give a few examples of compact sets and compact MSs.

Exercise 7 *In every* MS *every finite set is compact.*

Exercise 8 *Is the real axis* $(\mathbb{R}, |x - y|)$ *a compact* MS*?*

Exercise 9 *Which other intervals on the real axis besides* $[a, b]$ *are compact sets?*

Exercise 10 *Let* $X = [a, b] \times [c, d]$ *be a rectangle in the plane, that is, in the Euclidean space* $(\mathbb{R}^2, e_2)$. *Prove that* X *is compact.*

Exercise 11 *Let* (M, d) *be* MS *and* $A, B \subset M$. *Which of the following implications holds?*

$$A \text{ and } B \text{ are compact} \Rightarrow A \cup B \text{ is compact}$$
$$A \text{ and } B \text{ are compact} \Rightarrow A \cap B \text{ is compact}$$
$$A \subset B \text{ and } B \text{ is compact} \Rightarrow A \text{ is compact}$$
$$A \text{ and } B \text{ are compact} \Rightarrow A \setminus B \text{ is compact}$$

• *Extrema and compact sets.* We begin with continuous maps between MSs. Let (M, d) and (N, e) be MS. A map $f \colon M \to N$ is *continuous at the point* $a \in M$ if for every ε there is a δ such that

$$x \in M \wedge d(x, a) < \delta \Rightarrow e(f(x), f(a)) < \varepsilon$$

($\delta > 0$ is a real number). A map is *continuous* if it is continuous at every point of the space.

Exercise 12 *Let $f \colon M \to N$ be a map between metric spaces and $a \in M$. Prove* Heine's definition of continuity at a point:

$$f \text{ is continuous at } a \iff$$
$$\iff \forall (a_n) \subset M : \lim a_n = a \Rightarrow \lim f(a_n) = f(a).$$

Theorem 13 (extremal values) *Let (M, d) be MS, $f \colon M \to \mathbb{R}$ be a continuous function, and $X \subset M$ be a nonempty compact set. Then there exist $a, b \in X$ such that*

$$\forall x \in X : \ f(a) \le f(x) \le f(b).$$

Thus f attains on the set X both the smallest value $f(a)$ and the largest value $f(b)$.

Proof. We show that $f[X] = \{f(x) \colon \ x \in X\}$ is a bounded subset of $\mathbb{R}$. If $f[X]$ were not bounded from above, we could take a sequence $(a_n) \subset X$ with $\lim f(a_n) = +\infty$. By the assumption, (a_n) has a convergent subsequence (a_{m_n}) with

$$\lim a_{m_n} = a \in X.$$

By the continuity of f at a and Exercise 12, $\lim f(a_{m_n}) = f(a) \in \mathbb{R}$. This is a contradiction because $\lim f(a_{m_n}) = +\infty$. Boundedness of $f[X]$ from below follows in a similar way.

So we can define the real numbers $A = \inf(f[X])$ and $B = \sup(f[X])$. By the definition of infima, there is a sequence $(a_n) \subset X$ such that $\lim f(a_n) = A$. By the assumption, (a_n) has a convergent subsequence (a_{m_n}) with $\lim a_{m_n} = a \in X$. By the continuity of f at a and Exercise 12, $\lim f(a_{m_n}) = f(a)$. Subsequences preserve limits and $\lim f(a_{m_n}) = A$. Thus $f(a) = A$ and for every $x \in X$,

$$f(a) = A \leq f(x)$$

because $A = \inf(f[X])$. The other element $b \in X$ is obtained similarly. $\qquad\square$

- *Products of metric spaces.* For metric spaces (M, d) and (N, e), we define their *product* $(M \times N, d \times e)$ as follows. $M \times N$ is the Cartesian product of the sets M and N and the $d \times e$ metric on it is given by

$$(d \times e)((a_1, a_2), (b_1, b_2)) = \sqrt{d(a_1, b_1)^2 + e(a_2, b_2)^2} \ .$$

Exercise 14 *Prove that the product of two* MSs *is a* MS.

Exercise 15 *Prove that the product of two Euclidean* MSs

$$(\mathbb{R}^m, e_m) \quad and \quad (\mathbb{R}^n, e_n)$$

is (except for a formality in notation) the Euclidean MS

$$(\mathbb{R}^{m+n}, e_{m+n}) \ .$$

What is the "formality"?

- *Characterization of compact sets in Euclidean* MSs. We defined the ball $B(a, r)$ in a MS last time. A set $X \subset M$ in a MS (M, d) is *open* if

$$\forall\, a \in X \,\exists\, r : \ B(a, r) \subset X \ .$$

Here $r > 0$ is a real number, the radius of the ball $B(a, r)$. X is *closed* if $M \setminus X$ is open. X is *bounded* if

$$\exists\, a \in M \; \exists\, r : \; X \subset B(a, r) \,.$$

The *diameter* of the set X is, with $V = \{d(a, b) : \; a, b \in X\}$ ($\subset [0, +\infty)$), defined as

$$\mathrm{diam}(X) := \begin{cases} \sup(V) & \dots \quad \text{the set } V \text{ is bounded from above and} \\ +\infty & \dots \quad \text{the set } V \text{ is unbounded from above .} \end{cases}$$

Exercise 16 *Prove that any set X is bounded if and only if $\mathrm{diam}(X) < +\infty$.*

Exercise 17 *Prove that for every unbounded set X there exists a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) \geq 1$.*

In the following two exercises we review basic properties of open and closed sets in a MS.

Exercise 18 *Let (M, d) be a MS. Then the following holds.*

1. *The sets $\emptyset$ and M are both open and closed.*

2. *Any finite intersection of open subsets of M is an open set and any finite union of closed subsets of M is a closed set.*

3. *Any union of open subsets of M is an open set and any intersection of closed subsets of M is a closed set.*

Exercise 19 *Let (M, d) be a MS and $X \subset M$. Then*

$$\text{the set } X \text{ is closed } \iff$$
$$\iff \forall\, (a_n) \subset X \; \forall\, a \in M \left(\lim a_n = a \Rightarrow a \in X \right) .$$

Theorem 20 *Let (M, d) and (N, e) be* MS. *The following holds.*

1. *If $X \subset M$ is a compact set, then X is closed and bounded. The opposite implication does not hold, see Exercise 22.*

2. *If (M, d) and (N, e) are compact, then the product $(M \times N, d \times e)$ is compact too.*

Proof. 1. If X is not closed, then by Exercise 19 there exists a convergent sequence $(a_n) \subset X$ such that $\lim a_n = a \in M \setminus X$. This sequence does not have a convergent subsequence with limit in X, since each subsequence has limit a. When X is not bounded, we easily construct a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) > 1$ (Exercise 17). This sequence clearly has no convergent subsequence.

2. Let $(a_n) = ((a_{n,1}, a_{n,2}))$ be a sequence in the product MS. We choose a subsequence (b_n) such that $(b_{n,1})$ has a limit $b \in M$ in (M, d). From (b_n) we select a subsequence (c_n) such that $(c_{n,2})$ has a limit $c \in N$ in (N, e). It is not difficult to see that (c_n) is a subsequence of the sequence (a_n) and that it has in the product MS the limit

$$\lim c_n = (b, c) \in M \times N .$$

$\square$

Exercise 21 *Let (M, d) be a compact* MS *and $X \subset M$ be a closed set. Prove that X is compact.*

Exercise 22 *Let M be an infinite set and the metric d on it is given as $d(a, b) = 1$ for $a \neq b$ and $d(a, a) = 0$. Show that (M, d) is a* MS *that is bounded and closed but not compact.*

Theorem 23 (compact sets in $\mathbb{R}^n$) *In every Euclidean* MS *$(\mathbb{R}^n, e_n)$, a set $X \subset \mathbb{R}^n$ is compact if and only if it is bounded and closed.*

Proof. By the first part of the previous theorem, it suffices to prove that every bounded and closed set $X \subset \mathbb{R}^n$ is compact. From its boundedness it follows that for a real number $a > 0$,

$$X \subset K = [-a,\, a]^n = [-a,\, a] \times [-a,\, a] \times \cdots \times [-a,\, a] \subset \mathbb{R}^n\ .$$

The Euclidean MS (K, e_n) is compact by the Bolzano–Weierstrass theorem, part 2 of the previous theorem, and Exercise 15. Clearly, X is also closed in (K, e_n) (problem 24), so according to Exercise 21, X is compact in (K, e_n) and therefore in $(\mathbb{R}^n, e_n)$ (Exercise 25). $\square$

Exercise 24 *Let (M, d) be a* MS, *$A \subset B \subset M$ and A be a closed set in $(M, d) \Rightarrow A$ is closed also in the subspace (B, d).*

Exercise 25 *Let (M, d) be a* MS *and $A \subset B \subset M$. Then A is compact in $(M, d) \iff A$ is compact in the subspace (B, d).*

<div align="center">THANK YOU FOR YOUR ATTENTION</div>

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 5, 9, 11, 17 and 22.