MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 13 (May 18, 2026) SOME PARTICULAR DIFFERENTIAL EQUATIONS

• *Newton's law of force.* Differential equations (DE) are basic tools in mathematical models in physics, technology, biology, economics, etc. A basic example is *Newton's law of force*

$$mx'' = F\,,$$

where $x = x(t)$ ($\in \mathbb{R}$) is the position at the time t of a particle that has mass m and is exposed to the force F; we consider only the simple one-dimensional case. More generally, the force can be a function of time, position, and the velocity: $F = F(t, x, x')$. In the simplest situation F is constant, or slightly more generally F depends only on t—then $x(t) = \int\int F$. This is the case, for example, in Earth's gravitational field which does not change in time and (for small scales) does not depend on the position of the particle and certainly not on its speed. These are gross simplifications, especially the independence on x. Then the *equation of free fall* is

$$mx'' = -mg\,,$$

where g is the gravitational acceleration constant. All solutions are exactly the functions

$$\{x(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2\colon\ c_1,\, c_2 \in \mathbb{R}\}\,.$$

The two constants c_i express the fact that the movement of the falling particle is determined uniquely by the initial position $x(t_0)$ and the velocity $x'(t_0)$ at the time t_0.

Exercise 1 *Prove that the solutions of the equation of free fall are precisely these functions. Can we determine the movement of the particle uniquely by the position $x(t_0)$ and the velocity $x'(t_1)$ at different times t_0 and t_1?*

• The second example is the *equation of the radioactive decay*

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -kR\,.$$

It describes the evolution of the quantity $R = R(t)$ of decaying radioactive material in time t, where k is a material constant. Clearly, every function

$$R(t) = c\exp(-kt)\,,$$

where c is a constant, solves this equation.

The area of differential equations (DE) includes *ordinary differential equations* (ODE), which involve functions of just one variable, and *partial differential equations* (PDE) involving functions with several variables and their partial derivatives. The two previous equations are ODE. In the previous and this lecture, we confine ourselves to ODE.

• *Before we completely leave* PDE, we mention three important examples for the sake of interest: *Laplace's equation* (or the *equation of potential*)

$$u = u(x,\, y)\colon \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0\,,$$

the *diffusion equation* (or the *equation of heat conduction*)

$$u = u(x,\, t)\colon \quad \alpha^2 \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}\,,$$

and the *wave equation*

$$u = u(x,\, t): \quad a^2 \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\ ,$$

where α and a are constants. The physical meaning of these equations is apparent from their names.

• *The general form of an* ODE for the unknown function $y = y(x)$ is ($n \in \mathbb{N}$)

$$F(x,\, y,\, y',\, y'',\, \ldots,\, y^{(n)}) = 0\ ,$$

so that F is a function in $n+2$ variables. The *order of the equation* is the highest order of derivative occurring in the equation. The above equation of free fall is a second-order ODE, while the equation of radioactive decay is a first-order ODE.

An ODE ($n \in \mathbb{N}$)

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)\,,$$

where $a_i(x)$ and $b(x)$ are given functions and $y = y(x)$ is the unknown function, is the *linear differential equation* (with order n and the right side $b(x)$). If $b(x) = 0$, the equation is *homogeneous.*

ODE depending on some variables for the unknown function and its derivatives non-linearly are *nonlinear* differential equations. An example is the *pendulum equation*

$$\theta'' + (g/l) \cdot \sin\theta = 0$$

that describes the motion of a pendulum of length l swinging in a homogeneous gravitational field. In it, g denotes the constant of gravitational acceleration, and the angle $\theta = \theta(t)$ records the deviation of the pendulum from the vertical at the time t. For small

θ we have $\sin\theta \approx \theta$ and we can solve the linear approximation of the pendulum equation

$$\theta'' + (g/l) \cdot \theta = 0\,,$$

which is a linear ODE. The equations of free fall and of the radioactive decay are linear.

Exercise 2 *Try to guess some solution to the equation*

$$\theta'' + (g/l) \cdot \theta = 0\,.$$

• *Algebraic differential equations.* Differential equations ($n \in \mathbb{N}$)

$$F(x,\, y,\, y',\, y'',\, \ldots,\, y^{(n)}) = 0\,,$$

where F is a polynomial in $n+2$ variables, are *algebraic differential equations* (ADE). We present, without proofs, three results about ADE.

Recall that the *Euler gamma function* $\Gamma(z)$ is defined for complex z with $\mathrm{re}(z) > 0$ by the integral

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t}\,\mathrm{dt}\ .$$

Exercise 3 *Show that for every* $z \in \mathbb{C}$ *s* $\mathrm{re}(z) > 0$ *this integral converges. The integrand needs to be estimated both at* 0 *and at* $+\infty$*.*

Exercise 4 *Compute that* $\Gamma(1) = 1$ *and prove that* $\Gamma(z)$ *satisfies the functional equation*

$$\Gamma(z+1) = z\Gamma(z)\ .$$

Hint: integration by parts.

Thus $\Gamma(n+1)=n!$ for every $n\in\mathbb{N}_0$ and we see that the gamma function generalizes factorial. The following theorem is classical.

Theorem 5 (O. Hölder, 1887) *The function* $\Gamma(z)$ *does not satisfy any nontrivial* ADE, *for any nonzero complex polynomial* F *with* $n+2$ *variables.*

The second example involves the functions

$$\vartheta(z)=\sum_{n=0}^{\infty} z^{n^2} \ \text{ and } \ P(z)=\sum_{n=0}^{\infty} p(n)z^n=\prod_{n=1}^{\infty}\frac{1}{1-z^n}$$

($|z|<1$).

Exercise 6 *Prove that* $p(n)$ *is the number of ways to write* n *as a sum of natural numbers. Sums differing only in the order of summands are regarded as identical.*

Theorem 7 *The functions* $\vartheta(z)$ *and* $P(z)$ *satisfy (quite complicated)* ADE.

In the third example, we consider ADE for formal power series.

Exercise 8 *Let*

$$M(x)=\sum_{n=0}^{\infty} n!\cdot x^n=1+x+2x^2+6x^3+24x^4+\ldots .$$

Obtain for $M(x)$ *an* ADE, *in fact, a first-order linear* ODE.

We define

$$B(x)=\sum_{n=0}^{\infty} B_n x^n=\sum_{k=0}^{\infty}\frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}=1+\ldots$$

and ($k\in\mathbb{N}$)

$$\sum_{n=1}^{\infty} S(n,k)x^n=\frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}\,.$$

The numbers $S(n,k)$ ($\in\mathbb{N}_0$) are the *Stirling numbers* (of the second kind).

Exercise 9 *Show that $S(n, k)$ counts the set partitions of an n-element set with k blocks. Hint: the coefficient of x^n in the expansion of*

$$\frac{x^k}{(1-x)(1-2x)\ldots(1-kx)}$$

counts words u of length n over the alphabet $[k] = \{1, 2, \ldots, k\}$ such that every $i \in [k]$ occurs in u and that for every $i, j \in [k]$ with $i < j$ the first occurrence of i in u precedes that of j.

Thus we have

$$B_n = \sum_{k=1}^{n} S(n,\, k)$$

and B_n is the number of all set partitions of an n-element set. The numbers B_n are called *Bell numbers*.

Theorem 10 (M. Klazar, 2003) *The formal power series*

$$B(x) = \sum_{n=0}^{\infty} B_n x^n$$

does not satisfy any non-trivial ADE.

The proof method is similar to the proof of Hölder's theorem. One proves that no non-trivial ADE is compatible with the following "functional" equation for $B(x)$.

Exercise 11 *Using the above definition of $B(x)$, show that*

$$B(x) = 1 + \frac{x}{1-x} \cdot B(x/(1-x))\,.$$

Exercise 12 *Show that the exponential generating function of Bell numbers*

$$\sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1}$$

satisfies a nontrivial ADE.

• ODE *with separated variables.* This is the problem ($a, b \in \mathbb{R}$)

$$y(a) = b \wedge y' = f(x) \cdot g(y) \qquad \text{(SEP)}$$

for the unknown function $y = y(x)$, where $f(x)$ and $g(y)$ are defined and continuous on open intervals $I \ni a$ and $J \ni b$, respectively. We assume that $g \neq 0$ on J. We solve this problem locally uniquely by a function $y\colon I' \to J$, for an open interval I' satisfying $a \in I' \subset I$.

We transform the equation in the form

$$\tfrac{y'}{g(y)} = f(x)$$

and rewrite it using a primitive function $G = \int 1/g$ on J as

$$x \in I' \Rightarrow G(y(x))' = f(x)\,.$$

So we have the equation that for every $x \in I'$,

$$G(y(x)) = F(x) + c\,,$$

where $F = \int f$ is a given primitive function on I and c is a constant. So the solution $y(x)$ of the problem (SEP) is given as an implicit function in

$$\underbrace{G(y(x)) = F(x) + c}_{(*)} \ \text{ where } \ G = \int \tfrac{1}{g} \ \text{ and } \ F = \int F\,.$$

The constant c is determined by the relation $G(b) = F(a) + c$. It follows from the implicit function theorem that there exists an open interval I' and a unique function $y\colon I' \to J$ such that $a \in I' \subset I$, $y(a) = b$, and the relation $(*)$ holds on I'. We have obtained a unique solution to the problem (SEP) on I'.

Exercise 13 *Explain the use of the implicit function theorem. For example, why its assumptions are satisfied. Why is the solution of* (SEP) *locally unique, when the primitive functions G and F are not unique?*

Exercise 14 *Does the local uniqueness of the solution of* (SEP) *follow from Picard's theorem?*

• *Linear* ODE *of the first order.* By this we conclude our course. It is the problem ($x_0, y_0 \in \mathbb{R}$)

$$y(x_0) = y_0 \wedge y' + a(x)y = b(x) \tag{LIN}$$

for the unknown function $y = y(x)$. The functions $a(x)$ and $b(x)$ are given. They are defined and continuous on some open interval $I \ni x_0$.

Exercise 15 *Does the local uniqueness and existence of the solution to* (LIN) *follow from Picard's theorem?*

It in fact does, and we only have to solve the problem. First, we find a function $c = c(x)$, called an *integration factor*, such that

$$c \cdot (y' + ay) = (cy)' \, .$$

Then $cy' + acy = cy' + c'y$ and c must satisfy the equation $ac = c'$, that is, $(\log c)' = a$. The function $c = e^A$, where $A = \int a$, has the required property. We multiply the linear equation by the integration factor and get

$$(cy)' = \underbrace{c(y' + ay) = cb}_{c \cdot \text{(LIN)}} \, .$$

So $(cy)' = cb$ and $cy = D + c_0$, where $D = \int cb$ and c_0 is an integration constant. We have the solution $y = c^{-1}(D + c_0)$. To sum up,

$$y(x) = e^{-A(x)}\big(\int e^{A(x)}b(x) + c_0\big) \text{ where } A(x) = \int a(x)\,.$$

Note that $y(x)$ is defined on the whole I (the domain of definition of the functions a and b) and that the initial conditions $y(x_0) = y_0$ exactly correspond to the integration constants c_0.

Exercise 16 *Solve the equation with separated variables*

$$v \cdot v' = -\tfrac{gR^2}{(R+x)^2}\,.$$

Here $R \approx 6378\,\mathrm{km}$ *is the radius of the Earth,* $g \approx 9.81\,\mathrm{ms}^{-2}$ *is the gravitational acceleration,* $x > 0$ *is the height (in meters) of a particle that was ejected from Earth's surface with the speed* $v = v_0$*, and* $v = v(x)$ *is the speed of the particle in the height* x*. Find the* escape velocity*, also called the* second cosmic velocity*. It is the (minimum) velocity* v_0 *such that the particle will never fall back to Earth.*

Exercise 17 *Consider a particle with mass* m *that falls from the rest under the influence of constant gravity and on which, in addition to the weight, the resistance of the environment acts in such a way that the strength of the resistance is proportional to the speed of the particle. Find a first order linear* ODE *for this problem and solve it. Calculate the limit velocity that the particle (almost) reaches.*

THANK YOU FOR YOUR ATTENTION!

Here are twelve exam questions.

1. Define metric space and spherical metric. Prove that the hemisphere is not flat — T. 12 in L. 1.
2. Prove Ostrowski's theorem — T. 6 in L. 2.
3. Prove the Heine–Borel theorem — T. 11 in L. 3.
4. Prove the existence of n-th roots in $\mathbb{C}$ — T. 2 in L. 4.
5. Prove Baire's theorem — T. 19 in L. 4.
6. Explain, how to show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
— see L. 7.
7. Prove that the MS $C([0,1])$ of continuous functions (with the maximum metric) is complete — P. 17 in L. 6.
8. Prove the case $d = 2$ or the case $d = 3$ of Pólya's theorem — T. 8 in L 8.
9. Prove that $\rho \neq 0$ — T. 6 in L. 10.
10. Prove the Cauchy–Goursat theorem for rectangles — T. 12 in L. 10.
11. Prove Picard's theorem — T. 6 in L. 12.
12. Solve the differential equation $y' + ay = b$ for the unknown function $y = y(x)$ (and given functions $a(x)$ and $b(x)$) — see L. 13.