

MATHEMATICAL ANALYSIS 3 (NMAI056)

summer term 2025/26

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**LECTURE 12 (May 11, 2026) EXISTENCE THEOREMS
FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS:
PICARD'S AND PEANO'S**

• *One of the seven Millennium Problems* announced by the Clay Mathematical Institute in 2000 — by solving any of them one can earn 10^6 \$ — is the problem if there exists a smooth solution to the Navier–Stokes (partial differential) equations which describe flow of fluid in the (3-dim.) space.

• *Banach's fixed point theorem.* For Picard's theorem on differential equations, we will need two results about complete metric spaces, with which we therefore begin. The first of these is the well-known existence result of fixed points of a *contracting mapping* (*contraction*)

$$f: M \rightarrow M$$

of the metric space (M, d) into itself. This is any mapping such that for some constant $c \in (0, 1)$ for every $a, b \in M$,

$$d(f(a), f(b)) \leq c \cdot d(a, b).$$

Exercise 1 *Prove that every contracting mapping of a metric space into itself is continuous.*

Theorem 2 (Banach's on fixed point) *Every contraction*

$$f: M \rightarrow M$$

of a complete metric space has a unique fixed point: a point $a \in M$ such that

$$f(a) = a .$$

Furthermore, each sequence $(a_n) \subset M$ of iterations of f , where the point $a_1 \in M$ is arbitrary and $a_n = f(a_{n-1})$ for $n > 1$, converges to this fixed point a .

Proof. We show that any sequence $(a_n) \subset M$ of iterations of the function f is Cauchy. This can be seen immediately from the estimate that for every two indices $m > n$,

$$\begin{aligned} d(a_m, a_n) &\stackrel{\Delta\text{-inequality}}{\leq} \sum_{i=n}^{m-1} d(\underbrace{a_{i+1}}_{f(a_i)}, a_i) \\ &\stackrel{f \text{ is contr.}, \text{ def. of } a_i}{\leq} \sum_{i=n}^{m-1} c^{i-1} \cdot d(a_2, a_1) \\ &\stackrel{\text{adding } \geq 0 \text{ terms}}{\leq} d(a_2, a_1) \sum_{i=n}^{\infty} c^{i-1} \\ &\stackrel{\Sigma \text{ of geom. series}}{=} \frac{d(a_2, a_1) \cdot c^{n-1}}{1-c} \rightarrow 0, \quad n \rightarrow \infty . \end{aligned}$$

Since (M, d) is complete, we can define

$$a := \lim_{n \rightarrow \infty} a_n \in M .$$

Then due to the continuity of the function f (Exercise 1),

$$f(a) = f(\lim_{n \rightarrow \infty} a_n) = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} a_{n+1} = a$$

and a is a fixed point of f . Uniqueness follows from Exercise 3. \square

Exercise 3 Prove that the fixed point of any contraction of any metric space is unique.

Exercise 4 Prove Banach's Fixed Point Theorem under the weaker assumption that only some n -th iteration

$$f^{[n]} := \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times } f}: M \rightarrow M$$

of the mapping f of M into itself is contracting.

- *Completeness of a certain function space.* We need completeness of the following metric space.

Proposition 5 For every two real numbers $a < b$, the metric space

$$(C[a, b], d)$$

of continuous functions $f: [a, b] \rightarrow \mathbb{R}$ is complete with respect to the maximum metric

$$d(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)| .$$

Proof. This is Proposition 17 in lecture 6. □

- *Picard's theorem* guarantees the existence and uniqueness of solutions of first-order ordinary differential equations with explicit first derivative.

Theorem 6 (Picard's) Let $a, b \in \mathbb{R}$ and let $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous such that

$$|F(u, v) - F(u, w)| \leq M \cdot |v - w|$$

for every $u, v, w \in \mathbb{R}$ and some constant $M > 0$. Then there exists $\delta > 0$ and a unique function $f: [a - \delta, a + \delta] \rightarrow \mathbb{R}$ such that

$$f(a) = b \ \& \ \forall x \in [a - \delta, a + \delta]: \ f'(x) = F(x, f(x)) . \quad (1)$$

Proof. Let $I = [a - \delta, a + \delta]$, for some small $\delta > 0$ to be determined later. It is easy to see (Exercise 7) that f solves the equation (1) if and only if f solves the equation

$$\forall x \in I: f(x) = b + \int_a^x F(t, f(t)) dt. \quad (2)$$

We show that for any sufficiently small $\delta > 0$, the equation (2), and therefore also the equation (1), has on the interval I a unique solution f . The right side of the equation (2) defines a map

$$A: C(I) \rightarrow C(I), \quad A(f)(x) = g(x) = b + \int_a^x F(t, f(t)) dt.$$

We prove that A is a contraction of the metric space $(C(I), d)$ (with the maximum metric d). By Theorem 2 and Proposition 5, A has a unique fixed point, the function $f \in C(I)$ such that $A(f) = f$, and equations (1) and (2) have unique solutions. Let us prove that for any sufficiently small $\delta > 0$, the map A is a contraction. For any $f, g \in C(I)$ we have

$$\begin{aligned} d(A(f), A(g)) &= \\ \stackrel{\text{def. of } d}{=} & \max_{x \in I} |A(f)(x) - A(g)(x)| \\ \stackrel{\text{def. of } A}{=} & \max_{x \in I} \left| \int_a^x F(t, f(t)) dt - \int_a^x F(t, g(t)) dt \right| \\ \stackrel{\text{linearity of } \int}{=} & \max_{x \in I} \left| \int_a^x (F(t, f(t)) - F(t, g(t))) dt \right| \\ \stackrel{| \int h | \leq \int |h|}{\leq} & \max_{x \in I} \int_a^x |F(t, f(t)) - F(t, g(t))| dt \\ \stackrel{\text{ass. on } F, h \leq j \Rightarrow \int h \leq \int j}{\leq} & \max_{x \in I} \int_a^x M |f(t) - g(t)| dt \\ \stackrel{h \leq j \Rightarrow \int h \leq \int j}{\leq} & \max_{x \in I} \int_a^x M \cdot d(f, g) dt \\ \stackrel{\int_a^x c = (x - a)c}{=} & \delta M \cdot d(f, g). \end{aligned}$$

For example, if $\delta = 1/2M$ then A is a contraction with the constant $c = 1/2$. \square

Exercise 7 Prove that $f: I \rightarrow \mathbb{R}$ solves the equation (1) if and only if f solves the equation (2).

For example, the equation

$$f(1) = -3 \ \& \ f' = f$$

has a unique solution on a neighborhood of 1, because here $F(u, v) = v$ and the condition on the function F is satisfied with the constant $M = 1$. This solution is the function

$$f(t) = (-3/e) \exp(t) .$$

Exercise 8 Prove that Picard's theorem holds under weaker assumptions on F that there exist constants $h, M > 0$ such that

$$F: (a - h, a + h) \times (b - h, b + h) \rightarrow \mathbb{R}$$

is continuous and for every two pairs (u, v) and (u, w) from the definition domain F ,

$$|F(u, v) - F(u, w)| \leq M \cdot |v - w| .$$

• *Peano's Theorem* is another theorem on existence, but no longer uniqueness, of solutions of the above differential equations.

Theorem 9 (Peano's) Let $(a, b) \in U \subset \mathbb{R}^2$, where U is an open set, and let $F: U \rightarrow \mathbb{R}$ be continuous. Then there exists $\delta > 0$ and a function $f: [a - \delta, a + \delta] \rightarrow \mathbb{R}$ such that

$$f(a) = b \ \& \ \forall x \in [a - \delta, a + \delta]: \ f'(x) = F(x, f(x)) .$$

Proof. First, we note that it suffices to prove the version of Peano's theorem, let us call it HP (half-Peano), in which the interval $[a - \delta, a + \delta]$ is replaced by the interval $[a, a + \delta]$. Indeed, by HP there exists a $\delta' > 0$ and a function f_1 such that

$$f_1(-a) = b \text{ and } \forall t \in [-a, -a + \delta']: f_1'(t) = G(t, f_1(t)),$$

where $G(u, v) := -F(-u, v)$. Then for $f_2(t) := f_1(-t)$ we have $f_2(a) = b$ and for every $t \in [-\delta' + a, a]$,

$$f_2'(t) = -f_1'(-t) = -G(-t, f_1(-t)) = F(t, f_2(t)).$$

We combine this solution of our problem to the left of a with some of its solutions to the right of a , obtained again according to HP, and get a solution on a two-sided δ -neighborhood of the point a (Exercise 10).

So we prove HP: there exists $\delta > 0$ and a function $f: [a, a + \delta] \rightarrow \mathbb{R}$ such that

$$f(a) = b \wedge \forall t \in [a, a + \delta]: f'(t) = F(t, f(t)).$$

We take constants $a', b' > 0$ such that F is defined and continuous on $[a, a + a'] \times [b - b', b + b']$. So $|F| < L$ on this set, for some constant $L > 0$. Let's take the interval

$$I := [a, a + c], \text{ where } c := \min(a', b'/L),$$

and the set \mathcal{A} of functions

$$\{f: I \rightarrow \mathbb{R}: f(a) = b \wedge (s, t \in I \Rightarrow |f(s) - f(t)| \leq L|s - t|)\}.$$

According to the choice of c , for each $f \in \mathcal{A}$ the composite function $F(t, f(t))$, $t \in I$, is well-defined, continuous and bounded (by

the constant L). We can therefore define the functional $P: \mathcal{A} \rightarrow [0, +\infty)$,

$$P(f) := \max_{t \in I} \left| f(t) - b - \int_a^t F(s, f(s)) \, ds \right|.$$

It is easy to see as before that if $P(f) = 0$, then f is a solution of HP: $f(a) = b$ and $f'(t) = F(t, f(t))$ on $[a, a + c]$. By Theorem 14,

$$\mathcal{A} \subset C(I)$$

is a compact set in the metric space $(C(I), d)$ with the maximum metric. It is easy to see that the functional P is continuous (Exercise 11). Thus, it attains minimum value on some function $\varphi \in \mathcal{A}$.

We show that $P(\varphi) = 0$: we find functions $f_n \in \mathcal{A}$ for $n = 2, 3, \dots$ such that $P(f_n) \rightarrow 0$. We define them recursively as

$$\forall t \in [a, a + c/n]: f_n(t) := b$$

and

$$\forall t \in (a + c/n, a + c): f_n(t) := b + \int_a^{t-c/n} F(s, f_n(s)) \, ds.$$

It is not difficult to see that this recursion correctly and uniquely defines the function f_n and that $f_n \in \mathcal{A}$ (Exercise 12). Then for every $t \in [a, a + c/n]$ we have by the first part of the definition of f_n that

$$\left| f_n(t) - b - \int_a^t F(s, f_n(s)) \, ds \right| = \left| \int_a^t F(s, f_n(s)) \, ds \right| \leq \frac{Lc}{n},$$

and for every $t \in (a + c/n, a + c]$ by the second part of the definition of f_n , by the linearity of integrals, and by means of simple ML estimates that

$$\left| f_n(t) - b - \int_a^t F(s, f_n(s)) \, ds \right| = \left| \int_{t-c/n}^t F(s, f_n(s)) \, ds \right| \leq \frac{Lc}{n}.$$

Thus $0 \leq P(f_n) \leq \frac{Lc}{n}$ and indeed $P(f_n) \rightarrow 0$ for $n \rightarrow \infty$. \square

This proof is taken from: R. L. Pouso, Peano's Existence Theorem revisited, arXiv:1202.1152v1, 2012.

Exercise 10 *Explain in detail how the solution to the problem to the left of the point a can be combined with the solution to the right of point a and why these solutions can be combined.*

Exercise 11 *Prove that the functional P in the previous proof is continuous.*

Exercise 12 *Prove that the functions f_n in the previous proof are well defined and lie in the set \mathcal{A} .*

Exercise 13 (non-unique solutions) *For real numbers $a < 0 < b$ we define the function $f = f_{a,b}: \mathbb{R} \rightarrow \mathbb{R}$ as*

$$t \leq a \Rightarrow f(t) := (t - a)^3, \quad a \leq t \leq b \Rightarrow f(t) := 0$$

and

$$t \geq b \Rightarrow f(t) = (t - b)^3.$$

Prove that each of these functions is on \mathbb{R} a solution of the equation

$$f(0) = 0 \ \& \ f'(t) = 3f(t)^{2/3} := 3(f(t)^{1/3})^2.$$

The power $x^{1/3}$ is defined here for $x < 0$ as $-(-x)^{1/3}$.

Theorem 14 (Arzelà–Ascoli) *Let $I = [a, b]$ with $a < b$ and let $C(I)$ be the metric space of continuous functions $f: I \rightarrow \mathbb{R}$ with the maximum metric. Then $X \subset C(I)$ is compact if and only if two conditions hold.*

1. We have $|f(x)| \leq c$ for every $x \in I$, every $f \in X$ and some constant $c > 0$. We say that the functions in X are uniformly bounded.
2. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in X$ and every $x, y \in I$ we have $|x - y| \leq \delta \Rightarrow |f(x) - f(y)| \leq \varepsilon$. We say that the functions in X are uniformly uniformly continuous.

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Tuesday solutions to the Exercises 1, 3, 7 and 13. This is the last set of HWs.