MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 11 (April May 4, 2026) INTRODUCTION TO COMPLEX ANALYSIS 3

- *Integral $\int_{\partial R} f$ for $f\colon \mathbb{C}\setminus A\to\mathbb{C}$.* We derive properties of the integral $\int_{\partial R} f$ for holomorphic functions $f\colon \mathbb{C}\setminus A\to\mathbb{C}$, where $A\subset \mathrm{int}(R)$ is a compact set and R ($\subset\mathbb{C}$) is a rectangle.

Theorem 1 *Let R, A and functions $f,g\colon \mathbb{C}\setminus A\to\mathbb{C}$ be as above. The following holds.*

1. *For every $\alpha,\beta\in\mathbb{C}$,*

$$\textstyle\int_{\partial R}(\alpha f+\beta g)=\alpha\int_{\partial R} f+\beta\int_{\partial R} g\,.$$

2. *If $A=\{a\}$ and f is bounded on a deleted neighborhood of the point a, then*

$$\textstyle\int_{\partial R} f=0\,.$$

3. *If $a\in\mathbb{C}$ and $a\in\mathrm{int}(R)$, then*

$$\textstyle\int_{\partial R}\frac{1}{z-a}=\rho\,,$$

where ρ ($=2\pi i$) is the previously introduced constant.

Proof. 1. We proved this linearity earlier.

2. Let R_n be rectangles with $a\in\mathrm{int}(R_n)$ and $R_n\to a$. By ML bounds ($\mathrm{per}(R_n)\to 0$) also $\int_{\partial R_n} f\to 0$. Hence $\int_{\partial R} f=0$.

3. Let S be the square with the vertices $\pm 1\pm i$ and $R=a+S$ be the shifted copy. By the previous theorem, the definition of $\int_{\partial R}$

and the definition of ρ we have

$$\textstyle\int_{\partial R}\frac{1}{z-a}=\int_{\partial(a+S)}\frac{1}{z-a}=\int_{\partial S}\frac{1}{z}=\rho\,.$$

$\square$

Exercise 2 *Let* $a\in\mathrm{int}(R)$ *and* $k\in\mathbb{N}$ *with* $k\ge 2$*. Then*

$$\textstyle\int_{\partial R}\frac{1}{(z-a)^k}=0\,.$$

• *The Cauchy formula.* For simplicity we give it only for entire functions.

Theorem 3 (Cauchy formula) *Let* $f\colon\mathbb{C}\to\mathbb{C}$ *be entire,* $\rho=2\pi i$*,* R *be a rectangle and* $a\in\mathrm{int}(R)$*. Then*

$$\textstyle f(a)=\frac{1}{\rho}\int_{\partial R}\frac{f(z)}{z-a}\,.$$

Proof. Since $f'(a)$ exists, $\frac{f(z)-f(a)}{z-a}$ is bounded on a deleted neighborhood of the point a. By parts 1–3 of Theorem 1 we have that

$$\textstyle 0=\int_{\partial R}\frac{f(z)-f(a)}{z-a}=\int_{\partial R}\frac{f(z)}{z-a}-f(a)\int_{\partial R}\frac{1}{z-a}=\int_{\partial R}\frac{f(z)}{z-a}-f(a)\rho\,.$$

Since $\rho\ne 0$ (Theorem 6 in the last lecture), we immediately get the Cauchy formula. $\square$

A proof of Liouville's theorem. Let $f\colon\mathbb{C}\to\mathbb{C}$ be entire and bounded, so that $|f(z)|\le c$ for $z\in\mathbb{C}$ and a constant c (≥ 0). Let $a,b\in\mathbb{C}$ be two (different) points. By Exercise 4, for every large $s\in\mathbb{N}$ there is a square $S\subset\mathbb{C}$ with side length s such that $a,b\in\mathrm{int}(S)$ and for any $z\in\partial S$,

$$\textstyle |z-a|,\,|z-b|\ge\frac{s}{3}=\frac{\mathrm{per}(S)}{12}\,.$$

By the Cauchy formula and linearity of $\int_{\partial R}$,

$$f(a)-f(b)=\tfrac{1}{\rho}\int_{\partial S}\tfrac{f(z)}{z-a}-\tfrac{1}{\rho}\int_{\partial S}\tfrac{f(z)}{z-b}=\tfrac{a-b}{\rho}\int_{\partial S}\tfrac{f(z)}{(z-a)(z-b)}\,.$$

By the ML estimate, the last $\int$ is in absolute value at most

$$\tfrac{c}{\mathrm{per}(S)^2/144}\cdot\mathrm{per}(S)=\tfrac{144c}{4s}=\tfrac{36c}{s}\to 0\ \text{ for }\ s\to\infty\,.$$

Thus $f(a)=f(b)$ and f is constant. □

Exercise 4 *Let $a,b\in\mathbb{C}$. For every large $s\in\mathbb{N}$ there is a square $S\subset\mathbb{C}$ with side length s such that $a,b\in\mathrm{int}(S)$ and for any $z\in\partial S$ we have $|z-a|,|z-b|\ge\frac{s}{3}$.*

A proof that any entire function is analytic. Let $f\colon\mathbb{C}\to\mathbb{C}$ be entire, $a\in\mathbb{C}$ be arbitrary, and R be a rectangle such that $0,a\in\mathrm{int}(R)$ and for $z\in\partial R$ we have

$$\left|\tfrac{a}{z}\right|=\tfrac{|a|}{|z|}\le\tfrac{1}{2}\ \text{ and }\ |z-a|\ge 1$$

(Exercise 5). Let $m\in\mathbb{N}$. The Cauchy formula and the identity

$$\tfrac{1}{1-x}=1+x+x^2+\dots+x^m+\tfrac{x^{m+1}}{1-x}$$

imply

$$\begin{aligned}
f(a)&=\tfrac{1}{2\pi i}\int_{\partial R}\tfrac{f(z)}{z-a}\\
&=\tfrac{1}{2\pi i}\int_{\partial R}\tfrac{f(z)}{z}\big(\textstyle\sum_{n=0}^{m}\left(\tfrac{a}{z}\right)^n+\tfrac{(a/z)^{m+1}}{1-a/z}\big)\\
&=\textstyle\sum_{n=0}^{m}\big(\tfrac{1}{2\pi i}\int_{\partial R}\tfrac{f(z)}{z^{n+1}}\big)a^n+\tfrac{1}{2\pi i}\int_{\partial R}\tfrac{f(z)(a/z)^{m+1}}{z-a}\\
&=:\textstyle\sum_{n=0}^{m}c_na^n+\tfrac{1}{2\pi i}I_{m+1}\,.
\end{aligned}$$

By an ML estimate of the integral I_{m+1} we are done: for $m\to\infty$,

$$|I_{m+1}|\le\max\nolimits_{z\in\partial R}|f(z)|\cdot\tfrac{(1/2)^{m+1}}{1}\cdot\mathrm{per}(R)\to 0\,.$$

Hence for any $a \in \mathbb{C}$ we have

$$f(a) = \textstyle\sum_{n=0}^{\infty} c_n a^n, \ \text{ where } c_n = \frac{1}{2\pi i} \int_{\partial S} \frac{f(z)}{z^{n+1}}$$

and S is any rectangle with $0 \in \mathrm{int}(S)$. $\square$

Exercise 5 *For any $a \in \mathbb{C}$ there exists a rectangle R such that $0, a \in \mathrm{int}(R)$ and for any $z \in \partial R$ we have $|\frac{a}{z}| \le \frac{1}{2}$ and $|z-a| \ge 1$.*

- *Meromorphic functions and residues.* We generalize part 3 of Theorem 1. A set $A \subset \mathbb{C}$ is *discrete* if every open disc $B(z, r)$ ($\subset \mathbb{C}$) contains only finitely many elements of A.

Definition 6 (meromorphic functions) *We say that a holomorphic function*

$$f\colon U \setminus A \to \mathbb{C}\,,$$

where $A \subset U$ is a discrete set, is meromorphic with the set of poles A, if for every $a \in A$ we have on $B(a, r_a) \setminus \{a\}$ an expansion

$$f(z) = g_a(z) + \textstyle\sum_{j=1}^{k_a} c_{j,\,a} \cdot (z-a)^{-j}\,,$$

where $B(a, r_a) \subset U$, $B(a, r_a) \cap A = \{a\}$, $g_a\colon B(a, r_a) \to \mathbb{C}$ is holomorphic, $k_a \in \mathbb{N}$, and $c_{j,a} \in \mathbb{C}$ for $j = 1, 2, \ldots, k_a$ with $c_{j,k_a} \neq 0$.

The *residue of f at a* is the coefficient $c_{1,a}$. We denote it by $\mathrm{res}(f, a)$. By the Cauchy formula, $\mathrm{res}(f, a)$ is uniquely determined (Exercise 8).

Theorem 7 (residue theorem) *Let $f\colon U \setminus A \to \mathbb{C}$ be meromorphic with the set of poles A and let $R \subset U$ be a rectangle such that $\partial R \cap A = \emptyset$. Then*

$$\tfrac{1}{2\pi i} \int_{\partial R} f = \textstyle\sum_{a \in A \cap \mathrm{int}(R)} \mathrm{res}(f,\, a) = \sum_{a \in A \cap R} \mathrm{res}(f,\, a)$$

and both sums are finite.

Proof. If the intersection $A \cap R$ were infinite, we would contradict the discreteness of A (Exercise 9), and both sums are finite. For $a \in R \cap A$ we take mutually disjoint squares

$$S_a \subset \mathrm{int}(R) \cap B(a,\, r_a)$$

centered at a. We then divide the rectangle R into rectangles including all of these squares S_a and get that $\int_{\partial R}$ equals

$$\textstyle\sum_{a \in A \cap R} \int_{\partial S_a} f = \sum_{a \in A \cap R} \int_{\partial S_a} \left(g_a(z) + \sum_{j=1}^{k_a} \frac{c_{j,a}}{(z-a)^j} \right)$$

which equals $\sum_{a \in A \cap R} 2\pi i \cdot \mathrm{res}(f, a)$ and we are done. These equalities follow from Exercise 10, the definition of meromorphic functions, the linearity of integrals, the Cauchy–Goursat theorem, part 3 of Theorem 1 and Exercise 2. □

Exercise 8 *Why is the residue of f at a unique?*

Exercise 9 *Infinite subsets of rectangles always have at least one limit point.*

Exercise 10 *Partition any given rectangle R with prescribed disjoint rectangles $R_j \subset \mathrm{int}(R)$, $j = 1, 2, \ldots, k$, by lines into small rectangles including every R_j.*

Exercise 11 *What is the punch line of the following mathematical joke?*

> *Did you know that the contour integral of f around the border of France is zero? ??? All Poles are in Eastern Europe!*

• *Solution of the generalized Basel problem.* Let $k \in \mathbb{N}$. We obtain by means of the residue theorem a formula for the sum

$$\zeta(2k) = \textstyle\sum_{n=1}^{\infty} n^{-2k}\,.$$

Earlier we found by the Fourier analysis that $\zeta(2) = \frac{\pi^2}{6}$. Now we generalize it. We begin with two auxiliary results.

Proposition 12 *The function*

$$F(z) = 2\pi i \cdot \big(\mathrm{e}^{2\pi i z} - 1\big)^{-1} \colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$$

is meromorphic with the set of poles $\mathbb{Z}$. For every $n \in \mathbb{Z}$ we have $\mathrm{res}(F, n) = 1$*.*

Proof. The function $f(z) = \mathrm{e}^{2\pi i z} - 1$ is entire and $f(z) = 0 \iff z \in \mathbb{Z}$. For any $n \in \mathbb{Z}$ we have the expansion

$$f(z) = 2\pi i(z-n) + a_2(z-n)^2 + \dots$$

because $f'(n) = 2\pi i$. In a deleted neighborhood of $n \in \mathbb{Z}$ we have

$$\begin{aligned} F(z) &= \tfrac{2\pi i}{f(z)} = \tfrac{1}{z-n} \cdot \tfrac{1}{1 + (a_2/2\pi i)(z-n)) + \cdots} \\ &= (z-n)^{-1} + b_0 + b_1(z-n) + \cdots \end{aligned}$$

and $\mathrm{res}(F, n) = 1$. $\square$

Proposition 13 *Let $N \in \mathbb{N}$, $F(z)$ be as above and S_N be the square with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. Then for some $c \geq 0$ for every N and every $z \in \partial S_N$ we have*

$$|F(z)| \leq c\,.$$

Proof. Since $F(z) = 2\pi i \cdot (\mathrm{e}^{2\pi iz} - 1)^{-1}$, we need to cut off $\mathrm{e}^{2\pi iz} - 1$ from 0. For $z \in \partial S_N$ with $|\mathrm{im}(z)| \geq 1$ we have

$$\begin{aligned} \left|\mathrm{e}^{2\pi iz} - 1\right| &\geq \left|\left|\mathrm{e}^{2\pi iz}\right| - 1\right| = \left|\mathrm{e}^{\mathrm{re}(2\pi iz)} - 1\right| \\ &= \left|\mathrm{e}^{-2\pi\cdot\mathrm{im}(z)} - 1\right| \geq \min\left(1 - \mathrm{e}^{-2\pi},\, e^{2\pi} - 1\right) \\ &= 1 - \mathrm{e}^{-2\pi} > 0\,. \end{aligned}$$

For $z \in \partial S_N$ with $|\mathrm{im}(z)| \leq 1$ we use that $\mathrm{e}^{2\pi iz}$ is 1-periodic, and by reducing modulo 1 we work in the strip P given by $0 \leq \mathrm{re}(z) \leq 1$. Then $z = \frac{1}{2} + ix$, where $x \in \mathbb{R}$ with $|x| \leq 1$, and

$$\left|\mathrm{e}^{2\pi iz} - 1\right| = \left|\mathrm{e}^{\pi i}\mathrm{e}^{-2\pi x} - 1\right| = \left|\mathrm{e}^{-2\pi x} + 1\right| \geq 1 + \mathrm{e}^{-2\pi}\,.$$

Thus we can set $c = 2\pi \cdot (1 - \mathrm{e}^{-2\pi})^{-1}$. □

Theorem 14 *Let $k \in \mathbb{N}$. Then, with the fractions $\alpha_k \in \mathbb{Q}$ given at the end of the proof,*

$$\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k} = 1 + \tfrac{1}{2^{2k}} + \tfrac{1}{3^{2k}} + \tfrac{1}{4^{2k}} + \cdots = \alpha_k \cdot \pi^{2k}\,.$$

Proof. *Bernoulli numbers B_r* are the fractions defined in the expansion

$$\tfrac{x}{\mathrm{e}^x - 1} = \sum_{r=0}^{\infty} \tfrac{1}{r!} \cdot B_r \cdot x^r \quad (\in \mathbb{Q}[[x]])\,,$$

see Exercise 15. We take the earlier meromorphic function

$$F(z) = 2\pi i \cdot \left(\mathrm{e}^{2\pi iz} - 1\right)^{-1} \colon \mathbb{C} \setminus \mathbb{Z} \to \mathbb{C}$$

with the set of poles $\mathbb{Z}$ and residues $\mathrm{res}(F, n) = 1$, $n \in \mathbb{Z}$ (Proposition 12). For $f(z)$ holomorphic near $n \in \mathbb{Z}$ we have $\mathrm{res}(fF, n) = f(n)$ (Exercise 16). We set $f(z) = z^{-2k}$. For $N \in \mathbb{N}$ let S_N be the square with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. The residue theorem gives

$$\begin{aligned} \tfrac{1}{2\pi i} \int_{\partial S_N} \tfrac{F(z)}{z^{2k}} &= \sum_{n=-N}^{N} \mathrm{res}(F(z)z^{-2k},\, n) \\ &= \mathrm{res}(F(z)z^{-2k},\, 0) + 2\sum_{n=1}^{N} n^{-2k}\,. \end{aligned}$$

By Proposition 13 we have the bound $|F(z)| \le c$ for every $N \in \mathbb{N}$ and every $z \in \partial S_N$. By the ML estimate the above integral is in absolute value at most

$$\max_{z \in \partial S_N} \left|\tfrac{F(z)}{z^{2k}}\right| \cdot \mathrm{per}(S_N) \le \tfrac{c}{N^{2k}} \cdot (8N+4) \to 0 \ \text{ for } \ N \to \infty\,.$$

Hence

$$\textstyle\sum_{n=1}^{\infty} n^{-2k} = -\frac{1}{2} \cdot \mathrm{res}(F(z)z^{-2k},\, 0)\,.$$

By the definition of $F(z)$ and B_r,

$$F(z)z^{-2k} = 2\pi i z \cdot z^{-1-2k} \cdot \left(\mathrm{e}^{2\pi i z} - 1\right)^{-1} = \textstyle\sum_{r=0}^{\infty} \frac{B_r (2\pi i)^r z^{r-1-2k}}{r!}\,.$$

We set $r = 2k$ and get the residue

$$\mathrm{res}(F(z)z^{-2k},\, 0) = \tfrac{1}{(2k)!} \cdot (-1)^k B_{2k} (2\pi)^{2k}$$

and the sum

$$\textstyle\sum_{n=1}^{\infty} n^{-2k} = \frac{2^{2k-1}}{(2k)!}(-1)^{k+1} B_{2k} \cdot \pi^{2k} = \alpha_k \cdot \pi^{2k}\,,$$

with $\alpha_k = \frac{2^{2k-1}}{(2k)!}(-1)^{k+1} B_{2k}$ ($\in \mathbb{Q}$). □

Exercise 15 *Prove that the Bernoulli numbers $B_r \in \mathbb{Q}$.*

Exercise 16 *Prove that if $f(z)$ is holomorphic on a neighborhood of $n \in \mathbb{Z}$, then* $\mathrm{res}(fF, n) = f(n)$.

For $k \ge 2$, $B_{2k-1} = 0$ (Exercise 17). We have $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $\ldots$ (Exercise 18). This proof is taken from the book P. D. Lax and L. Zalcman, *Complex Proofs of Real Theorems*, AMS (The American Mathematical Society), Providence, RI (Rhodes Island), 2012. For more about complex analysis in Czech see J. Veselý, *Komplexní analýza pro učitele*, Karolinum, Praha, 2000.

Exercise 17 *Show that* $B_3 = B_5 = B_7 = \cdots = 0$.

Exercise 18 *Deduce a recurrence for* B_r *and check the above values of* B_0, B_1, B_2, B_4 *and* B_6.

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 4, 5, 11 and 15.