MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
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LECTURE 10 (April 23, 2025) INTRODUCTION TO COMPLEX ANALYSIS 2

• *The Cauchy–Goursat theorem for rectangles.* It is the main result of this lecture:

$$\int_{\partial R} f = 0 \text{ whenever } R \subset U \text{ and } f\colon U \to \mathbb{C} \text{ is holomorphic}\,.$$

• *Linear functions.* We begin with a simple proof of the C.–G. theorem for rectangles in the case that the function $f(z)$ is linear. Let $k \in \mathbb{N}$ and $u = ab$ ($\subset \mathbb{C}$) be a segment. The *k-equipartition* of u is the partition of u into k subsegments of the same length $\frac{1}{k}|u|$. It is the image of the partition $0 < \frac{1}{k} < \frac{2}{k} < \dots < \frac{k-1}{k} < 1$ of the interval $I = [0, 1]$.

Exercise 1 *Let $a, b, \alpha, \beta \in \mathbb{C}$ with $a \neq b$. Prove from the definition of the Cauchy integral that*

$$\int_{ab}(\alpha z + \beta) = \alpha\big(\tfrac{b^2}{2} - \tfrac{a^2}{2}\big) + \beta(b - a) = g(b) - g(a)\,,$$

where $g(z) = \frac{1}{2}\alpha z^2 + \beta z$. Hint — use equipartitions of ab.

Corollary 2 (the easy C.–G. theorem) *Let $\alpha, \beta \in \mathbb{C}$ and $R \subset \mathbb{C}$ be a rectangle. Then*

$$\int_{\partial R}(\alpha z + \beta) = 0\,.$$

Proof. Let a, b, c, d be the canonical vertices of R and $f(z) = \alpha z + \beta$. By the definition of $\int_{\partial R}$ and the previous exercise, $\int_{\partial R} f$ is $g(b) - g(a) + g(c) - g(b) + g(d) - g(c) + g(a) - g(d) = 0$. $\square$

• *Contour integration.* We express the Cauchy integral $\int_u f$ in terms of real Riemann integrals.

Exercise 3 *Prove the next proposition.*

Proposition 4 ($\int_u$ via Riemann $\int$) *Let $a, b \in \mathbb{C}$, $a \neq b$, $f\colon ab \to \mathbb{C}$ be continuous and $\varphi(t) = t(b-a) + a\colon [0,1] \to \mathbb{C}$ be the parametrization of the segment $u = ab$. Then*

$$\begin{aligned} \textstyle\int_u f &= \textstyle\int_0^1 f(\varphi(t)) \cdot \varphi'(t)\,\mathrm{dt} = (b-a)\int_0^1 f(\varphi(t))\,\mathrm{dt} \\ &= (b-a)\big(\textstyle\int_0^1 \mathrm{re}\big(f(\varphi(t))\big)\,\mathrm{dt} + i \cdot \int_0^1 \mathrm{im}\big(f(\varphi(t))\big)\,\mathrm{dt}\big) \end{aligned}$$

– except for the first $\int$, all other are Riemann $\int$s.

To relate our approach to the standard one, we define the contour integral $\int_\varphi f$. Let $f\colon U \to \mathbb{C}$ and $\varphi\colon [a,b] \to U$ be a continuous and piece-wise smooth function. The *integral of the function f along the curve φ* is given by

$$\begin{aligned} \textstyle\int_\varphi f &= \textstyle\int_a^b f(\varphi(t)) \cdot \varphi'(t)\,\mathrm{dt} \\ &= \textstyle\int_a^b \mathrm{re}\big(f(\varphi(t)) \cdot \varphi'(t)\big)\,\mathrm{dt} + i \cdot \int_a^b \mathrm{im}\big(f(\varphi(t)) \cdot \varphi'(t)\big)\,\mathrm{dt}\,, \end{aligned}$$

if the last two real Riemann integrals exist. By Proposition 4, the Cauchy integral $\int_u$ is a particular case of $\int_\varphi$.

Exercise 5 *Let $\varphi(t) = \mathrm{e}^{2\pi i t}\colon [0, \frac{1}{2}] \to \mathbb{C}$ be given by*

$$\varphi(t) = \textstyle\sum_{n=0}^{\infty} \frac{(2\pi i t)^n}{n!}\,.$$

It is a parametrization of the upper unit semicircle. Let $f(z) = z^2$. Find $\int_\varphi f$.

• *The constant ρ ($= 2\pi i$).* If the constant ρ defined in the next theorem were 0, the complex analysis would collapse.

Theorem 6 **(ρ)** *Let S be the square with the vertices $\pm 1 \pm i$. Then $\rho = \int_{\partial S} \frac{1}{z} \neq 0$, in fact $\mathrm{im}(\rho) \geq 4$.*

Proof. The canonical vertices of S are $a = -1 - i$, $b = 1 - i$, $c = 1 + i$ and $d = -1 + i$. Let

$$p_n = (a_0,\, a_1,\, \dots,\, a_n)$$

be the n-equipartition of ab. Multiplication by i counter-clockwisely rotates p_n around the origin by $\frac{1}{2}\pi$:

$$q_n = ip_n = (ia_0,\, ia_1,\, \dots,\, ia_n)$$

is the n-equipartition of bc. Similarly, $r_n = iq_n = -p_n$ is the n-equipartition of cd and $s_n = ir_n = -ip_n$ is the n-equipartition of da. But for $f(z) = \frac{1}{z}$ we have

$$C(f,\, p_n) = C(f,\, q_n) = C(f,\, r_n) = C(f,\, s_n)\,.$$

Indeed, extending the fraction by i gives

$$\begin{aligned} C(f,\, p_n) &= \textstyle\sum_{j=1}^{n} \frac{(b-a)/n}{a+j(b-a)/n} = \sum_{j=1}^{n} \frac{(ib-ia)/n}{ia+j(ib-ia)/n} \\ &= \textstyle\sum_{j=1}^{n} \frac{(c-b)/n}{b+j(c-b)/n} = C(f,\, q_n) \end{aligned}$$

and similarly for the other two equalities. As $b - a = 2$ and $a = -1 - i$, extending the fraction by $\frac{2j}{n} - 1 + i$ we get

$$\begin{aligned} \mathrm{im}(C(f, p_n)) &= \mathrm{im}\big(\textstyle\sum_{j=1}^{n} \frac{2/n}{-1-i+2j/n}\big) = \mathrm{im}\left(\frac{2}{n}\sum_{j=1}^{n} \frac{2j/n-1+i}{(2j/n-1)^2+1}\right) \\ &= \textstyle\frac{2}{n}\sum_{j=1}^{n} \frac{1}{(2j/n-1)^2+1} \geq \frac{2}{n}\sum_{j=1}^{n} \frac{1}{2} = 1\,. \end{aligned}$$

By Exercise 7,

$$\mathrm{im}\big(\textstyle\int_{\partial S} \frac{1}{z}\big) = 4 \cdot \mathrm{im}\left(\int_{ab} \frac{1}{z}\right) = 4 \cdot \lim_{n\to\infty} \mathrm{im}(C(\frac{1}{z},\, p_n)) \geq 4 \cdot 1 = 4\,.$$

Indeed $\rho \neq 0$. □

Exercise 7 *Let* (z_n) *be a convergent sequence of complex numbers. Prove that* $\mathrm{im}(\lim\, z_n) = \lim \mathrm{im}(z_n)$*.*

Exercise 8 *The previous proof gives that* $\mathrm{re}(\rho) = 0$*.*

Exercise 9 **($\rho = 2\pi i$)** *Again, let* $a = -1-i$ *and* $b = 1-i$*. Compute by Proposition 4 that* $\int_{ab} \frac{1}{z} = \frac{1}{2}\pi i$*. Thus, by the previous proof,*

$$\rho = 4 \cdot \tfrac{1}{2}\pi i = 2\pi i\,.$$

Exercise 10 *Let* $\varphi(t)\colon [0, 1] \to \mathbb{C}$*,* $\varphi(t) = \mathrm{e}^{2\pi i t}$ *and* $f(z) = \frac{1}{z}$*. Show that* $\int_\varphi f = 2\pi i$*.*

• *Nested sets and quarters.* Recall that a set $X \subset \mathbb{C}$ has *diameter*

$$\mathrm{diam}(X) = \sup(\{|x - y|\colon\ x,\, y \in X\})\ \ (\in [0,\, +\infty) \cup \{+\infty\})\,.$$

Exercise 11 *Let* $A_n \neq \emptyset$ *be closed sets such that* $\mathbb{C} \supset A_1 \supset A_2 \supset \dots$ *and* $\lim \mathrm{diam}(A_n) = 0$*. Then*

$$\textstyle\bigcap_{n=1}^{\infty} A_n \neq \emptyset\,.$$

Hint – see the proof of Baire's theorem.

Let R be a rectangle with the canonical vertices (a, b, c, d). We define the *quarters* of R. Let $e = \frac{a+b}{2}$, $f = \frac{b+c}{2}$, $g = \frac{c+d}{2}$ and $h = \frac{d+a}{2}$ be the midpoints of the sides of R and $j = \frac{a+c}{2}$ be the center point of R. The four *quarters* of R are the rectangles A, B, C and D with the canonical vertices, respectively,

$$(a,\, e,\, j,\, h),\, (e,\, b,\, f,\, j),\, (j,\, f,\, c,\, g)\ \text{ and }\ (h,\, j,\, g,\, d)\,.$$

So R is divided into four quarters by cutting along the segments eg and hf. For each quarter E we have

$$\mathrm{per}(E) = \tfrac{1}{2} \cdot \mathrm{per}(R)\ \text{ and }\ \mathrm{diam}(E) = \tfrac{1}{2} \cdot \mathrm{diam}(R)\,.$$

• *The Cauchy–Goursat theorem for rectangles.* This is the next theorem, which we already stated at the beginning.

Theorem 12 (the C.–G. theorem for R) *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $R \subset U$ be a rectangle. Then*

$$\textstyle\int_{\partial R} f = 0\,.$$

Proof. Let f, U and R be as stated. We define rectangles

$$R = R_0 \supset R_1 \supset R_2 \supset \dots$$

such that always R_{n+1} is a quarter of R_n and

$$\textstyle\big|\int_{\partial R_{n+1}} f\big| \ge \frac{1}{4}\big|\int_{\partial R_n} f\big|\,. \tag{1}$$

Suppose that R_0, R_1, $\dots$, R_n have been already defined. Let A, B, C and D be the quarters of the rectangle R_n. We claim that

$$\textstyle\int_{\partial R_n} f = \int_{\partial A} f + \int_{\partial B} f + \int_{\partial C} f + \int_{\partial D} f\,. \tag{2}$$

This follows from parts 3 and 4 of the last theorem in the previous lecture. We express each of the integrals $\int_{\partial A} f$, $\int_{\partial B} f$, $\int_{\partial C} f$ and $\int_{\partial D} f$ as the sum of four integrals over the sides. On the right-hand side of equality (2) we get 16 terms. Eight of them correspond to the sides of quarters lying inside R_n. They mutually cancel out because they form four pairs of opposite orientations of the same segment. The remaining eight terms correspond to the sides of the quarters lying on ∂R_n. These add to the integral on the left-hand side of equality (2). Inequalities (1) follow from the triangle inequality: for some quarter $E \in \{A, B, C, D\}$ we have

$$\textstyle\big|\int_{\partial E} f\big| \ge \frac{1}{4}\cdot\big|\int_{\partial R_n} f\big|$$

and can set $R_{n+1} = E$.

By Exercise 11 there exists a point z_0 such that

$$z_0 \in \bigcap_{n=0}^{\infty} R_n\,.$$

Since $R_0 = R \subset U$, we have $z_0 \in U$. Let an $\varepsilon > 0$ be given. Since $f'(z_0)$ exists, there is a $\delta > 0$ and a function $\Delta\colon\, B(z_0, \delta) \to \mathbb{C}$ such that $B(z_0, \delta) \subset U$, and that on $B(z_0, \delta)$ we have $|\Delta(z)| \le \varepsilon$ and

$$f(z) = \underbrace{f(z_0) + f'(z_0)\cdot(z - z_0)}_{g(z)} + \underbrace{\Delta(z)\cdot(z - z_0)}_{h(z)}\ .$$

We consider functions $g(z)$ and $h(z)$. It is clear that $g(z)$ is linear and $h(z)$ $(= f(z) - g(z))$ is continuous on $B(z_0, \delta)$. Let $n \in \mathbb{N}_0$ be so large that $R_n \subset B(z_0, \delta)$ – only here we need that $\lim \operatorname{diam}(R_n) = 0$, for the existence of the point z_0 it is not essential, see Exercise 14. By the linearity of the integral $\int_{\partial R}$ and Corollary 2 we have that

$$\int_{\partial R_n} f = \int_{\partial R_n} g + \int_{\partial R_n} h \overset{\text{Cor. 2}}{=} \int_{\partial R_n} h\,. \tag{3}$$

Thus

$$\begin{aligned} \Big|\int_{\partial R_n} h\Big| &\overset{\text{ML estimate}}{\le} \max_{z\in\partial R_n} |\Delta(z)\cdot(z - z_0)| \cdot \operatorname{per}(R_n) \\ &\le \varepsilon \cdot \operatorname{diam}(R_n)\cdot \operatorname{per}(R_n) \\ &= \varepsilon \cdot \frac{\operatorname{diam}(R)}{2^n} \cdot \frac{\operatorname{per}(R)}{2^n} \\ &\le \varepsilon \cdot \frac{\operatorname{per}(R)^2}{4^n}\,. \end{aligned} \tag{4}$$

We used that quartering halves diameters and perimeters, and that the diameter of a rectangle is smaller than its perimeter. According to the previous results we have that

$$\tfrac{1}{4^n}\Big|\int_{\partial R} f\Big| \overset{\text{ineq. (1)}}{\le} \Big|\int_{\partial R_n} f\Big| \overset{\text{eq. (3)}}{=} \Big|\int_{\partial R_n} h\Big| \overset{\text{ineq. (4)}}{\le} \varepsilon \cdot \tfrac{\operatorname{per}(R)^2}{4^n}\,.$$

Hence $|\int_{\partial R} f| \le \varepsilon \cdot \mathrm{per}(R)^2$. It holds for every $\varepsilon > 0$, and we have $\int_{\partial R} f = 0$. □

Exercise 13 *What is the value of the function* $\Delta(z)$ *in the proof at* $z = z_0$*?*

Exercise 14 *Prove that for non-emptiness of the intersection in Exercise 11, it suffices to assume that the set* A_1 *is bounded (instead of the zero limit of diameters). Show that it does not hold in general metric spaces.*

The theorem originates with the French mathematician *Augustin-Louis Cauchy (1789–1857)*. For several years he lived in Prague in political exile. Cauchy assumed continuity of the derivative f'. It was *Édouard Goursat (1858–1936)* who proved the theorem in 1900 only with the assumption of existence f', in the article E. Goursat, Sur la definition générale des fonctions analytiques, d'après Cauchy, *Trans. Amer. Math. Soc.* **1** (1900), 14–16.

The C.–G theorem for rectangles suffices for our purposes. The theorem holds for general curves and we only outline the proof.

Theorem 15 (Cauchy–Goursat) *Let* $f\colon U \to \mathbb{C}$ *be a holomorphic function and* $\varphi\colon [a,b] \to U$ *be a continuous and piecewise smooth function. Suppose that* φ *is injective, except that* $\varphi(a) = \varphi(b)$*, and that the interior of* φ *— the bounded connected component of* $\mathbb{C} \setminus \varphi[\,[a,b]\,]$ *— is a subset of* U*. Then*

$$\int_{\varphi} f = 0\,.$$

Sketch of the proof. We draw in $\mathbb{C}$, with the help of horizontal and vertical lines, a fine square grid $\mathcal{M}$. A simple closed curve

$\psi\colon [a,b] \to U$ runs along the sides of the grid $\mathcal{M}$ and satisfies that (i) for a given $\varepsilon > 0$ it holds that $|\int_\varphi f - \int_\psi f| \le \varepsilon$ (the curve ψ closely approximates the curve φ) and (ii) the interior of the curve ψ is a subset of the set U. Then

$$\int_\psi f = \sum_{R \in M} \int_{\partial R} f = \sum_{R \in M} 0 = 0\,,$$

where M is the set of elementary rectangles of the grid $\mathcal{M}$ lying inside the curve ψ. The first equality holds for the same reason as equality (2) and the first equality in (5) below. The second equality follows from (ii) and the preceding C.–G. theorems for rectangles. By (i), $|\int_\varphi f| \le \varepsilon$. This is true for every ε and $\int_\varphi f = 0$. $\square$

Exercise 16 *As in* Exercise 10, *let* $\varphi(t)\colon [0,1] \to \mathbb{C}$, $\varphi(t) = \mathrm{e}^{2\pi i t}$ *(now we parameterize the whole unit circle) and* $f(z) = z^k$, *where* $k \in \mathbb{Z} \setminus \{-1\}$. *Show that*

$$\int_\varphi f = 0\,.$$

This does not follow from the C.–G. theorem!

• *Independence of* $\int_{\partial R}$ *on* R. We show that in some situations the integral $\int_{\partial R} f$ does not depend much on the rectangle R. Recall that compact sets $A \subset \mathbb{C}$ are exactly the closed and bounded sets.

Proposition 17 (independence of $\int_{\partial R} f$ on R) *Let* $R, S \subset \mathbb{C}$ *be rectangles and let* $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ *be a compact. Let* $f\colon \mathbb{C} \setminus A \to \mathbb{C}$ *be a holomorphic function. Then*

$$\int_{\partial R} f = \int_{\partial S} f\,.$$

Proof. Let A, R, S, and f be as given, and let first $S \subset \mathrm{int}(R)$. By extending the sides of S, we divide R into nine rectangles R_1,

$R_2, \ldots, R_8, S$. Then indeed

$$\int_{\partial R} f \overset{\text{as in (2)}}{=} \sum_{j=1}^{8} \int_{\partial R_j} f + \int_{\partial S} f \overset{\text{Thm. 12, } R_j \subset \mathbb{C} \setminus A}{=} \int_{\partial S} f \, . \qquad (5)$$

We reduce the case of general rectangles R and S to the previous case. By Exercises 18 and 19, for any rectangles R and S and any nonempty compact set $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ there is a rectangle T such that

$$A \subset \mathrm{int}(T) \ \text{ and } \ T \subset \mathrm{int}(R) \cap \mathrm{int}(S) \, .$$

By the already proven case,

$$\int_{\partial R} f = \int_{\partial T} f = \int_{\partial S} f \, .$$

$\square$

Exercise 18 *Prove that every nonempty intersection of two rectangles is a rectangle.*

Exercise 19 *Prove that for every rectangles R and S and every nonempty compact set $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ there is a rectangle T such that*

$$A \subset \mathrm{int}(T) \ \text{ and } \ T \subset \mathrm{int}(R) \cap \mathrm{int}(S) \, .$$

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 5, 9, 16 and 19.