EXERCISES FOR TUTORIAL 5 OF MA 2, Nov 9, 2023

In the following $M = (M, d)$ is a metric space.

1. Define what it means that the space M is complete and show that every compact space is complete.
2. Give (and justify) an example of a space M that is complete but not compact.
3. Let $M = ([0, 1), |x - y|)$. Give (and justify) an example of a continuous and bounded function $f\colon M \to \mathbb{R}$ that has no maximum on M.
4. Let $F(x, y) = x^2 + 2y^2 - 1$. For which points $(x_0, y_0) \in \mathbb{R}^2$ with $F(x_0, y_0) = 0$ is the assumption of the theorem on implicit functions (TIF) satisfied, so that we can solve the equation $F(x, y) = 0$ for $y = f(x)$ in a neighborhood of x_0? Compute $f'(x_0)$ in two ways: using the formula in TIF and then directly (find $f(x)$ explicitly and differentiate it).
5. The same for the variable y, that is, for the function $x = g(y)$.