EXERCISES FOR TUTORIAL 11 OF MA 2, Dec 21, 2023

On multivariate Riemann integrals. We define for an (n-dimensional compact) interval $J := [a_1, b_1] \times \cdots \times [a_n, b_n]$ its volume as

$$\mathrm{vol}(J) := \prod_{j=1}^{n} (b_j - a_j) \ .$$

For a partition $P = (P_1, \ldots, P_n)$ of J, where $P_j = (a_j = t_{0,j} < t_{1,j} < \cdots < t_{m_j,j} = b_j)$ with $m_j \in \mathbb{N}$ is a partition of the interval $[a_j, b_j]$, we call any interval

$$[t_{i_1-1,1},\, t_{i_1,1}] \times [t_{i_2-1,2},\, t_{i_2,2}] \times \cdots \times [t_{i_n-1,n},\, t_{i_n,n}] \ ,$$

where $1 \le i_j \le m_j$ for $j = 1, 2, \ldots, n$, the *little interval (determined by the partition P)*.

1. How many little intervals determined by the partition P are there?
2. Prove that $\mathrm{vol}(J) = \sum \mathrm{vol}(I)$, where we sum over all little intervals I determined by the partition P.
3. What does one mean by a subdivision (or a refinement) of the partition P?
4. Write precisely Fubini's theorem for bivariate functions $f(x, y)$.
5. Compute in both orders of variables the two-dimensional Riemann integral

   $$\int_J f(x, y)$$

   for $J = [1, 2] \times [2, 3]$ and $f(x, y) = x \sin(x + y)$.