LECTURE 9, 4/13/2022

TAYLOR POLYNOMIALS AND SERIES. PRIMITIVES

• *An announcement.* In lecture 7 I simplified and extended the definition of derivative of a function at a point — limit points suffice — and I modified lectures 7 and 8 accordingly, lecture eighth has hardly changed. Today's lecture culminates with a complete proof of the existence of a primitive function of any continuous function. Before that we discuss Taylor polynomials and series.

• *Taylor polynomials.* In Lecture 7, after defining derivative, we learned that differentiability of a function $f\colon M \to \mathbb{R}$ at a limit point $a \in M \subset \mathbb{R}$ of M provides the linear approximation

$$f(x) = f(a) + f'(a) \cdot (x - a) + o(x - a) \ \ (x \to a)\ .$$

In the following theorem, which is also a definition, we use higher-order derivatives to strengthen the approximation by means of polynomials.

Theorem 1 (Taylor polynomial) *Let $n \in \mathbb{N}_0$ and let $f\colon U(b, \delta) \to \mathbb{R}$ be a function with finite $f^{(n)}(b) \in \mathbb{R}$. For $n = 0$, this means that f is continuous at b. Then there is exactly one polynomial*

$$p(x) := \sum_{j=0}^{n} a_j (x - b)^j,\ a_j \in \mathbb{R},\ s.\ t.\ \underbrace{\lim_{x \to b} \frac{f(x) - p(x)}{(x - b)^n} = 0}_{(0)}\ .$$

Its coefficients are given by the formula $a_j = f^{(j)}(b)/j!$. We call it the Taylor polynomial of the function f of order n centered at b and denote it as $T_n^{f,b}(x)$.

$T_n^{f,b}(x)$ equals

$$f(b)+f'(b)\cdot(x-b)+\frac{f''(b)}{2}\cdot(x-b)^2+\cdots+\frac{f^{(n)}(b)}{n!}\cdot(x-b)^n$$

and the linear approximation above is $T_1^{f,a}(x)$. Also, $T_0^{f,b}(x)=f(b)$ and for every $n\in\mathbb{N}$ we have the identity

$$\big(T_n^{f,\,b}(x)\big)'=T_{n-1}^{f',\,b}(x)\ .$$

To prove Theorem 1 we need the following lemma.

Lemma 2 (on the zero polynomial) *For any numbers $b\in\mathbb{R}$ and $n\in\mathbb{N}_0$ and any polynomial $p(x)=\sum_{j=0}^{n}a_jx^n$ with $a_j\in\mathbb{R}$ one has the implication*

$$\lim_{x\to b}\frac{p(x)}{(x-b)^n}=0\Rightarrow\forall\,j=0,\,1,\,\ldots,\,n\colon a_j=0\ .$$

Proof. Induction on n. For $n=0$ it is true, $a_0/1\to0$ gives $a_0=0$. Let $n>0$ and let the limit in the hypothesis of the implication hold. Then $p(b)=\lim_{x\to b}p(x)=0$. Thus b is a root of $p(x)$ and $p(x)=(x-b)\cdot q(x)$, where $q(x)$ is a real polynomial of degree at most $n-1$. But

$$0=\lim_{x\to b}\frac{p(x)}{(x-b)^n}=\lim_{x\to b}\frac{q(x)}{(x-b)^{n-1}}\ ,$$

and by induction $q(x)$ is the zero polynomial. Hence so is $p(x)$. $\square$

Proof of Theorem 1. The assumption on $f^{(n)}(b)$ means that (after possibly decreasing δ) for every $j=0,1,\ldots,n-1$ there exists $f^{(j)}\colon U(b,\delta)\to\mathbb{R}$. First we prove that for $p(x)=T_n^{f,b}(x)$ the limit (0) holds. For $n=0$ this follows from the continuity of f

at b. For $n = 1$ we have by the theorem on arithmetic of limits of functions that the limit

$$\lim_{x \to b} \frac{f(x) - \overbrace{(f(b) + f'(b) \cdot (x - b))}^{T_1^{f,\,b}(x)}}{x - b} = \lim_{x \to b} \frac{f(x) - f(b)}{x - b} - \lim_{x \to b} f'(b)$$

indeed equals $f'(b) - f'(b) = 0$. For $n \geq 2$ we get by l'Hospital's rule, the identity above and induction on n that

$$\begin{aligned} \lim_{x \to b} \frac{f(x) - T_n^{f,\,b}(x)}{(x - b)^n} &= \lim_{x \to b} \frac{\big(f(x) - T_n^{f,\,b}(x)\big)'}{((x - b)^n)'} \\ &= (1/n) \lim_{x \to b} \frac{f'(x) - T_{n-1}^{f',\,b}(x)}{(x - b)^{n-1}} \\ &= (1/n) \cdot 0 = 0 \ . \end{aligned}$$

Let $p(x) = \sum_{j=0}^{n} b_j x^j$ with $b_j \in \mathbb{R}$ be any polynomial for which the limit (0) holds. Then

$$\begin{aligned} \lim_{x \to b} \frac{p(x) - T_n^{f,\,b}(x)}{(x - b)^n} &= \lim_{x \to b} \frac{p(x) - f(x)}{(x - b)^n} + \lim_{x \to b} \frac{f(x) - T_n^{f,\,b}(x)}{(x - b)^n} \\ &= 0 + 0 = 0 \ . \end{aligned}$$

Thus, according to the previous lemma, $p(x) = T_n^{f,b}(x)$. $\square$

We state concisely the strengthened approximation.

Corollary 3 (Taylor approximation) *If $n \in \mathbb{N}_0$ and $f\colon U(b,\delta) \to \mathbb{R}$ is a function with finite $f^{(n)}(b) \in \mathbb{R}$ (i.e., f is continuous at b for $n = 0$), then for $x \in U(b,\delta)$ and $x \to b$,*

$$f(x) = T_n^{f,b}(x) + o((x-b)^n) = \sum_{j=0}^{n} \frac{f^{(j)}(b)}{j!}(x-b)^j + \underbrace{o((x-b)^n)}_{e(x)} \ .$$

The notation $o(\dots)$ means that $\lim_{x\to b} e(x)/(x-b)^n = 0$.

• *Taylor polynomials of elementary functions.* We present several Taylor polynomials centered at 0. We justify these formulas, calculate a few limits with them, and discuss when the extension of Taylor polynomials of f to an infinite series converges to $f(x)$. In the following formulas $n \in \mathbb{N}_0$ is arbitrary.

1. $f(x) = \exp x$ has TP $T_n^{f,0}(x) = \sum_{j=0}^n x^j/j!$.
2. $f(x) = \sin x$ has TP $T_{2n+1}^{f,0}(x) = \sum_{j=0}^n (-1)^j x^{2j+1}/(2j+1)!$.
3. $f(x) = \cos x$ has TP $T_{2n}^{f,0}(x) = \sum_{j=0}^n (-1)^j x^{2j}/(2j)!$.
4. For $\forall\, a \in \mathbb{R}$, $f(x) = (1+x)^a$ has TP $T_n^{f,0}(x) = \sum_{j=0}^n \binom{a}{j} x^j$. Here
$$\binom{a}{j} = a(a-1)(a-2)\dots(a-j+1)/j!\ ,$$
with $\binom{a}{0} := 1$, is the *generalized binomial coefficient*.
5. $f(x) = \log(1+x)$ has TP $T_n^{f,0}(x) = \sum_{j=1}^n (-1)^{j+1} x^j/j$ for $n > 0$ and $T_0^{f,0}(x) = 0$.
6. $f(x) = \log\left(\frac{1}{1-x}\right)$ has TP $T_n^{f,0}(x) = \sum_{j=1}^n x^j/j$ for $n > 0$ and $T_0^{f,0}(x) = 0$.

7. $f(x) = \arctan x$, the inverse tangent, has TP $T^{f,0}_{2n+1}(x) = \sum_{j=0}^{n}(-1)^j x^{2j+1}/(2j+1)$.
8. $f(x) = \arcsin x$, i.e., the inverse sine, has TP $T^{f,0}_{2n+1}(x) = \sum_{j=0}^{n}\binom{j-1/2}{j}x^{2j+1}/(2j+1)$.
9. $f(x) = \arccos x$, the inverse cosine, has TP $T^{f,0}_{2n+1}(x) = \pi/2 - \sum_{j=0}^{n}\binom{j-1/2}{j}x^{2j+1}/(2j+1)$.

Proof of Formula 1. Clearly, $\exp^{(j)}(x) = \exp(x)$ for every $j \in \mathbb{N}_0$ and $\exp(0) = 1$. $\square$

Proof of Formula 2. Clearly, $\sin^{(j)}(x) = \sin x$ for $j \equiv 0 \pmod 4$, $\sin^{(j)}(x) = \cos x$ for $j \equiv 1 \pmod 4$, $\sin^{(j)}(x) = -\sin x$ for $j \equiv 2, \pmod 4$, $\sin^{(j)}(x) = -\cos x$ for $j \equiv 3 \pmod 4$ and $\sin 0 = 0$ and $\cos 0 = 1$. $\square$

Proof of Formula 3. Similarly to the previous derivation. $\square$

Proof of Formula 4. For any $x \in (-1, 1)$, any $j \in \mathbb{N}_0$ and any $a \in \mathbb{R}$,

$$((1+x)^a)^{(j)} = a(a-1)\dots(a-j+1)(1+x)^{a-j} ,$$

with $((1+x)^a)^{(0)} = (1+x)^a$. Clearly, $(1+0)^{a-j} = 1$. $\square$

Proof of Formula 5. For any $x \in (-1, 1)$ and any $j \in \mathbb{N}$,

$$\begin{aligned}(\log(1+x))^{(j)} &= (-1)(-2)\dots(-j+1)\cdot(1+x)^{-j} \\ &= (-1)^{j+1}(j-1)!\cdot(1+x)^{-j}\end{aligned}$$

and $(\log(1+x))^{(0)} = \log(1+x)$. Clearly, $\log(1+0) = 0$ and $(1+0)^{-j} = 1$. $\square$

Proof of Formula 6. It follows from the previous formula because on $(-1,1)$, $\log\left(\frac{1}{1-x}\right) = -\log(1+(-x))$. $\square$

Proof of Formula 7. $(\arctan x)^{(0)} = \arctan x$ and

$$(\arctan x)^{(1)} = \frac{1}{1+x^2} = \frac{1}{2i}\left(\frac{1}{x-i} - \frac{1}{x+i}\right) .$$

Thus for every $j \in \mathbb{N}$,

$$(\arctan x)^{(j)} = -\frac{i}{2} \cdot (-1)^{j-1}(j-1)! \left((x-i)^{-j} - (x+i)^{-j}\right) .$$

Also, $(\arctan x)^{(0)}(0) = 0$ and for any even $j \geq 2$ we have that $(\arctan x)^{(j)}(0) = 0$ too. For every odd $j \in \mathbb{N}$,

$$\begin{aligned}(\arctan x)^{(j)}(0) &= \frac{i}{2} \cdot \underbrace{(-1)^{j-1}}_{=i^{2j-2}}(j-1)! \cdot 2 \cdot i^{-j} \\ &= i^{j-1}(j-1)! = (-1)^{(j-1)/2}(j-1)! .\end{aligned}$$

$\square$

However, we differentiated complex functions of the real variable here. We therefore derive this Taylor polynomial again without using $\mathbb{C}$.

Proposition 4 (TP of f from TP of f') *We suppose that $f\colon U(0,\delta) \to \mathbb{R}$ has finite $f'\colon U(0,\delta) \to \mathbb{R}$ and finite $f^{(n+1)}(0) \in \mathbb{R}$, $n \in \mathbb{N}_0$. Then for $x \to 0$,*

$$f'(x) = \sum_{j=0}^{n} a_j x^j + o(x^n), \quad a_j \in \mathbb{R} ,$$

$$\Rightarrow f(x) = f(0) + \sum_{j=0}^{n} \frac{a_j}{j+1} \cdot x^{j+1} + o(x^{n+1}) .$$

Proof. We work with center 0. By Theorem 1 on the uniqueness of TP, it follows from the hypothesis of the implication that for $j = 0, 1, \ldots, n$, $a_j = f^{(j+1)}(0)/j!$. By the same theorem, the coefficient of x^{j+1} in the TP of the function f is equal to

$$\frac{f^{(j+1)}(0)}{(j+1)!} = \frac{a_j}{j+1}\ .$$

$\square$

Thus the TP of the function $\arctan x$ is obtained from the TP $T_{2n}^{f,0}(x) = \sum_{j=0}^{n}(-1)^j x^{2j}$ of the derivative $\arctan'(x) = \frac{1}{1+x^2}$. We get this TP from (partial sums of) the geometric series $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots$, $x \in (-1, 1)$. $\square$

Proof of Formula 8. This follows immediately from the TP of $\arcsin'(x) = (1-x^2)^{-1/2}$, Proposition 4 and formula 4. $\square$

Proof of Formula 9. Proceed as in the previous derivation. $\square$

• *Computing limits by Taylor polynomials.* We will use Corollary 3. Using $T_1^{f,0}$ in Formula 2, we immediately see that

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{x+o(x)}{x} = \lim_{x\to 0}\frac{x}{x} + \lim_{x\to 0}\frac{o(x)}{x} = 1 + 0 = 1\ .$$

Or, using $T_2^{f,0}$ in Formula 3,

$$\begin{aligned}\lim_{x\to 0}\frac{x^4}{(\cos x - 1)^2} &= \lim_{x\to 0}\frac{x^4}{(1 - x^2/2 - 1 + o(x^2))^2} \\ &= \lim_{x\to 0}\frac{x^4}{x^4/4 + o(x^4)} = \lim_{x\to 0}\frac{1}{\frac{1}{4} + o(x^4)/x^4} = 4\ .\end{aligned}$$

V. I. Arnold's booklet *Gjujgens i Barrou, N'juton i Guk* (Nauka, Moskva 1989) mentions on p. 21 the problem

$$\lim_{x\to 0}\frac{\sin(\tan x)-\tan(\sin x)}{\arcsin(\arctan x)-\arctan(\arcsin x)}=?$$

Taylor polynomials may not be the best approach here (but why?). See `https://kam.mff.cuni.cz/~klazar/ArnoldLimEng.pdf`, if and when I write it down.

• *Taylor series.* Taylor series of a function arises from its Taylor polynomials by extending them to infinity.

Definition 5 (Taylor series) *Let $f\colon U(a,\delta)\to\mathbb{R}$ have finite $f^{(n)}\colon U(a,\delta)\to\mathbb{R}$ for every $n\in\mathbb{N}_0$. If for every $x\in U(a,\delta)$,*

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}\cdot(x-a)^n\ ,$$

we say that the function f is on $U(a,\delta)$ the sum of its Taylor series $\sum_{n=0}^{\infty}f^{(n)}(a)\cdot(x-a)^n/n!$ centered at a.

Hence Taylor polynomials are partial sums of Taylor series. The following theorem shows when the situation of the previous definition occurs. For $n\in\mathbb{N}_0$ and a function $f\colon U(a,\delta)\to\mathbb{R}$ with finite $f^{(n)}(a)\in\mathbb{R}$, we define the *remainder of the Taylor polynomial* $T_n^{f,a}(x)$ as

$$R_n^{f,\,a}(x):=f(x)-T_n^{f,\,a}(x),\ x\in U(a,\,\delta)\ .$$

Theorem 6 (remainders of TP) *Suppose that $n \in \mathbb{N}_0$ and that $f\colon U(a,\delta) \to \mathbb{R}$ with finite $f^{(n+1)}\colon U(a,\delta) \to \mathbb{R}$. Then the following holds.*

1. *(Lagrange's remainder) $\forall\, x \in P(a,\delta)\ \exists\, c$ between a and x such that*
$$R_n^{f,\,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} \cdot (x-a)^{n+1}\ .$$

2. *(Cauchy's remainder) $\forall\, x \in P(a,\delta)\ \exists\, c$ between a and x such that*
$$R_n^{f,\,a}(x) = \frac{f^{(n+1)}(c) \cdot (x-c)^n}{n!} \cdot (x-a)\ .$$

Proof. We prove more generally that for any $g\colon U(a,\delta) \to \mathbb{R}$ with finite and nonzero $g'\colon U(a,\delta) \to \mathbb{R}$ and any $x \in P(a,\delta)$ there is a number c between a and x such that
$$R_n^{f,\,a}(x) = \frac{1}{n!} \cdot \frac{g(x)-g(a)}{g'(c)} \cdot f^{(n+1)}(c) \cdot (x-c)^n\ . \tag{R}$$

Then Lagrange's remainder arises for $g(t) := (x-t)^{n+1}$ and Cauchy's for $g(t) := t$.

Let $x \in P(a,\delta)$ and the function g be as stated. Consider the auxiliary function
$$F(t) := f(x) - \sum_{i=0}^{n} \frac{f^{(i)}(t)}{i!} \cdot (x-t)^i\ .$$

We apply to F, g and the interval I with endpoints a and x Cauchy's mean value theorem. On this interval F is continuous,

$F(x) = 0$, $F(a) = f(x) - T_n^{f,a}(x)$, $g(a) \neq g(x)$ (due to Lagrange's mean value theorem) and on I,

$$\begin{aligned} F'(t) &= -f'(t) - \sum_{i=1}^{n} \left(\frac{f^{(i+1)}(t)}{i!} \cdot (x-t)^i - \frac{f^{(i)}(t)}{i!} \cdot i(x-t)^{i-1} \right) \\ &= -\frac{f^{(n+1)}(t)}{n!} \cdot (x-t)^n \ . \end{aligned}$$

By Cauchy's mean value theorem (equality $(*)$) there exists a number $c \in I^0$ such that

$$-\frac{f(x) - T_n^{f,a}(x)}{g(x) - g(a)} = \frac{F(x) - F(a)}{g(x) - g(a)} \overset{(*)}{=} \frac{F'(c)}{g'(c)} = -\frac{f^{(n+1)}(c) \cdot (x-c)^n}{n! \cdot g'(c)} \ .$$

Now the relation (R) follows by a simple rearrangement. $\square$

For all nine formulas for TP above we state for which $x \in \mathbb{R}$ they give Taylor series of f centered at 0 and converging to $f(x)$. We omit the proofs, they follow easily from the previous theorem.

1. $\forall\, x \in \mathbb{R}$, $\mathrm{e}^x = \sum_{n \geq 0} x^n/n!$.
2. $\forall\, x \in \mathbb{R}$, $\sin x = \sum_{n \geq 0} (-1)^n x^{2n+1}/(2n+1)!$.
3. $\forall\, x \in \mathbb{R}$, $\cos x = \sum_{n \geq 0} (-1)^n x^{2n}/(2n)!$.
4. $\forall\, x \in (-1, 1)$ and $\forall\, a \in \mathbb{R}$, $(1+x)^a = \sum_{n \geq 0} \binom{a}{n} x^n$.
5. $\forall\, x \in (-1, 1)$, $\log(1+x) = \sum_{n \geq 1} (-1)^{n+1} x^n/n$.
6. $\forall\, x \in (-1, 1)$, $\log\left(\frac{1}{1-x}\right) = \sum_{n \geq 1} x^n/n$.
7. $\forall\, x \in (-1, 1)$, $\arctan x = \sum_{n \geq 0} (-1)^n x^{2n+1}/(2n+1)$.
8. $\forall\, x \in (-1, 1)$, $\arcsin x = \sum_{n \geq 0} \binom{n-1/2}{n} x^{2n+1}/(2n+1)$.

9. $\forall\, x \in (-1,1)$, $\arccos x = \frac{\pi}{2} - \sum_{n\ge 0}\binom{n-1/2}{n}x^{2n+1}/(2n+1)$.

Some of these expansions hold in larger domains. Expansion 4 with $a \in \mathbb{N}_0$ holds $\forall\, x \in \mathbb{R}$, expansion 5 holds also for $x = 1$, expansion 6 also for $x = -1$, expansion 7 also for $x = 1$ and expansions 8 and 9 also for $x = -1$.

Coefficients in Taylor series can often be interpreted combinatorially. We give without proof one example of many.

Proposition 7 (Bell numbers B_n) *For any $x \in (-1,1)$ it is true that*

$$\mathrm{e}^{\mathrm{e}^x-1} = \exp(\exp(x)-1) = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$$

where B_n is the number of partitions of an n-element set.

For example, $B_3 = 5$ because of the five partitions $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{1\},\{2,3\}\}$ and $\{\{1\},\{2\},\{3\}\}$ of the set $\{1,2,3\}$.

• *Primitive functions.* An interval $I \subset \mathbb{R}$ is *non-trivial* if $I \ne \emptyset, \{a\}$ for every $a \in \mathbb{R}$. Non-trivial are exactly those non-empty intervals, each point of which is their limit point.

Definition 8 (primitives) *For any functions* $F, f\colon I \to \mathbb{R}$ *defined on a non-trivial interval* $I \subset \mathbb{R}$*, we say that* F *is a primitive (function) of* f*, and write* $F = \int f$*, if* F *has finite derivative on* I *and*

$$\forall\, b \in I\colon\; F'(b) = f(b)\;.$$

Sometimes F *is also called an antiderivative of* f*.*

We emphasize that for every $b \in I$, including endpoints, here $F'(b)$ always means the ordinary, two-sided derivative; other texts on analysis often differ in this because they define ordinary derivative only in interior points. It follows from the earlier result on derivatives that every primitive function is continuous. For example, $ax^2/2 + bx + c$ is a primitive of the linear function $ax + b$ on any nontrivial interval, e^x is on $\mathbb{R}$ a primitive of itself, $c + \arcsin x$ is on $(-1, 1)$ an antiderivative of the function $1/\sqrt{1-x^2}$ and $2x^{3/2}/3$ is a PF of $\sqrt{x}$ on $[0, +\infty)$.

Antiderivative of a given function is not determined uniquely, but every two of them differ only by a constant shift.

Theorem 9 (non-uniqueness of PF) $F_1, F_2, f\colon I \to \mathbb{R}$ *are functions defined on a nontrivial interval* $I \subset \mathbb{R}$ *and both* F_1 *and* F_2 *are primitives of* f*. Then there is a* $c \in \mathbb{R}$ *such that*

$$F_1 - F_2 = c \text{ on } I\;.$$

Conversely, if F *is a primitive of* f *then for every* $c \in \mathbb{R}$ *also* $F + c$ *is a primitive of* f*.*

Proof. Let F_1, F_2, f and I be, as stated, and $a < b$ be any two numbers from I. By Lagrange's mean value theorem, used for the function $F_1 - F_2$ and the interval $[a, b]$, there exists a $c \in (a, b)$ such that

$$\begin{aligned}\frac{(F_1 - F_2)(b) - (F_1 - F_2)(a)}{b - a} &= (F_1 - F_2)'(c) = F_1'(c) - F_2'(c) \\ &= f(c) - f(c) = 0 \;.\end{aligned}$$

So $F_1(b) - F_2(b) = F_1(a) - F_2(a)$ and $F_1(x) - F_2(x) = c$ for some constant c and every $x \in I$.

The second claim is clear, $(F + c)' = F' + c' = f + 0 = f$. $\square$

In the rest of the lecture we prove that every continuous function has an antiderivative. We have to prepare for it some tools.

• *Exchange of limits and derivatives.* We prove a theorem describing situations when one can swap limit for $n \to \infty$ and differentiation, without changing the result. We use this theorem below to prove Theorem 16 on existence of antiderivatives. But first we have to introduce pointwise and uniform convergence and prove the Moore–Osgood theorem.

Definition 10 ($f_n \to f$) *$M \subset \mathbb{R}$ is a set and $f, f_n \colon M \to \mathbb{R}$ for $n \in \mathbb{N}$ are functions. When*

$$\forall\, \varepsilon\; \forall\, x \in M\; \exists\, n_0 \colon n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon \;,$$

we write $f_n \to f$ (on M) and say that the functions f_n converge on M pointwisely to f.

Thus for every $x \in M$, $\lim f_n(x) = f(x)$.

Definition 11 ($f_n \rightrightarrows f$) *$M \subset \mathbb{R}$ is a set and $f, f_n \colon M \to \mathbb{R}$ for $n \in \mathbb{N}$ are functions. When*

$$\forall\, \varepsilon\, \exists\, n_0\, \forall\, x \in M \colon n \geq n_0 \Rightarrow |f_n(x) - f(x)| < \varepsilon\ ,$$

we write $f_n \rightrightarrows f$ (on M) and say that the functions f_n converge on M uniformly to f.

Now one requires more: single n_0 works for every $x \in M$. Clearly, $f_n \rightrightarrows f$ implies that $f_n \to f$, but the converse in general does not hold.

The following theorem is also called the Moore–Osgood theorem.

Theorem 12 (swapping limits) *Let $f_n, f \colon M \to \mathbb{R}$, where $n \in \mathbb{N}$ and $M \subset \mathbb{R}$, let $f_n \rightrightarrows f$ (on M), $A \in \mathbb{R}^*$ be a limit point of M and let $\lim_{x \to A} f_n(x) =: a_n \in \mathbb{R}$ for every n. Then the following finite limits exist and are equal:*

$$\lim a_n = \lim_{x \to A} f(x), \text{ i.e., } \lim_{n \to \infty} \lim_{x \to A} f_n(x) = \lim_{x \to A} \lim_{n \to \infty} f_n(x)\ .$$

Proof. From $f_n \rightrightarrows f$ (on M) it follows that $(f_n(x)) \subset \mathbb{R}$ is uniformly Cauchy for $x \in M$, that is, for every ε there is an n_0 such that for every $x \in M$ and every $m, n \geq n_0$,

$$|f_m(x) - f_m(x)| < \varepsilon\ .$$

Then for every two fixed indices $m, n \geq n_0$ the limit transition $\lim_{x \to A}$ gives the inequality $|a_m - a_n| \leq \varepsilon$. Thus $(a_n) \subset \mathbb{R}$ is a Cauchy sequence and has a finite limit $\lim a_n =: a \in \mathbb{R}$. The next estimate holds for every $n \in \mathbb{N}$ and every $x \in M$:

$$|f(x) - a| \le \underbrace{|f(x) - f_n(x)|}_{V_1} + \underbrace{|f_n(x) - a_n|}_{V_2} + \underbrace{|a_n - a|}_{V_3} \ .$$

Let an ε be given. Because $\lim a_n = a$, there exists an n_0 such that $n \ge n_0 \Rightarrow V_3 < \varepsilon/3$. Because $f_n \rightrightarrows f$ (on M), there exists an n_1 such that $n \ge n_1 \Rightarrow V_1 < \varepsilon/3$ for every $x \in M$. Let $m \ge \max(n_0, n_1)$. Since $\lim_{x \to A} f_m(x) = a_m$, we can take a δ such that $V_2 < \varepsilon/3$ for $n := m$ and every $x \in P(A, \delta) \cap M$. Thus for $n := m$ and every $x \in P(A, \delta) \cap M$,

$$|f(x) - a| \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

and $\lim_{x \to A} f(x) = a = \lim a_n$. $\square$

Here is the theorem that swaps limits and derivatives.

Theorem 13 (swapping df/dx and $\lim_{n\to\infty}$) *For $n \in \mathbb{N}$ let $f_n\colon I \to \mathbb{R}$ be functions defined on a nontrivial interval $I \subset \mathbb{R}$ and such that the following three conditions hold.*

1. *For every n there exists $f_n'\colon I \to \mathbb{R}$.*
2. *$f_n' \rightrightarrows f$ (on I) for some function $f\colon I \to \mathbb{R}$.*
3. *There exists an $a \in I$ such that the sequence $(f_n(a)) \subset \mathbb{R}$ converges.*

Then $f_n \to F$ (on I) for some function $F\colon I \to \mathbb{R}$, there exists $F'\colon I \to \mathbb{R}$ and

$$F' = f \text{ on } I, \text{ i.e., } \big(\lim_{n\to\infty} f_n\big)' = \lim_{n\to\infty} f_n' \ .$$

Proof. Let f_n, I, f and a be, as stated, and let $b \in I$ be any point. First we prove that the sequence $(f_n(b)) \subset \mathbb{R}$ is Cauchy. For $b = a$ this is true by Condition 3, so we can assume that for example $a < b$, the case with $b < a$ is treated similarly. Let an ε be given. It follows from Conditions 2 and 3 that the sequence of functions (f_n') is uniformly Cauchy on I and that the sequence $(f_n(a))$ is Cauchy. So there exists an n_0 such that $m, n \geq n_0 \Rightarrow |f_m'(x) - f_n'(x)| < \varepsilon$ for every $x \in I$ and also $m, n \geq n_0 \Rightarrow |f_m(a) - f_n(a)| < \varepsilon$. We take two arbitrary indices $m, n \geq n_0$ and use Lagrange's mean value theorem for the function $f_m - f_n$ and the interval $[a, b]$. This gives for some number $c \in (a, b)$ the equality and estimate

$$\frac{(f_m - f_n)(b) - (f_m - f_n)(a)}{b - a} = (f_m - f_n)'(c)$$

and

$$\begin{aligned} |f_m(b) - f_n(b)| &\leq |b - a| \cdot |f_m'(c) - f_n'(c)| + |f_m(a) - f_n(a)| \\ &< (b - a)\varepsilon + \varepsilon = \varepsilon(b - a + 1)\ , \end{aligned}$$

respectively. So the sequence $(f_n(b))$ is Cauchy, therefore convergent, and for every $b \in I$ we can define

$$F(b) := \lim f_n(b) \in \mathbb{R}\ .$$

So we get the function $F\colon I \to \mathbb{R}$ such that $f_n \to F$ (on I).

We prove that $F' = f$ on I. We use the previous theorem and then verify that its assumptions are satisfied. For any $b \in I$ indeed

$$\begin{aligned} F'(b) &= \lim_{x \to b} \frac{F(x) - F(b)}{x - b} \\ &= \lim_{x \to b} \lim_{n \to \infty} \frac{f_n(x) - f_n(b)}{x - b} \\ &\overset{\text{Theorem 12}}{=} \lim_{n \to \infty} \lim_{x \to b} \frac{f_n(x) - f_n(b)}{x - b} \\ &= \lim_{n \to \infty} f_n'(b) = f(b)\ . \end{aligned}$$

We check that in this use of Theorem 12 its assumptions are satisfied. We use the theorem for the sequence of functions

$$g_n(x) := \frac{f_n(x) - f_n(b)}{x - b} \colon I \setminus \{b\} \to \mathbb{R}\ .$$

Of course, $\lim_{x \to b} g_n(x) = f_n'(b)$ for every n and also $\lim f_n'(b) = f(b)$. It remains to check that $g_n \rightrightarrows g$ (on $I \setminus \{b\}$) for the function

$$g(x) := \frac{F(x) - F(b)}{x - b}\ .$$

For this we check that the sequence $(g_n(x))$ is uniformly Cauchy on $I \setminus \{b\}$. For every $m, n \in \mathbb{N}$ and every $x \in I \setminus \{b\}$ we have the identity

$$\begin{aligned} |g_m(x) - g_n(x)| &= \frac{|(f_m(x) - f_n(x)) - (f_m(b) - f_n(b))|}{|x - b|} \\ &\overset{(*)}{=} \frac{\cancel{|x - b|} \cdot |f_m'(c) - f_n'(c)|}{\cancel{|x - b|}} \\ &= \underbrace{|f_m'(c) - f_n'(c)|}_{V},\ \text{for a } c \text{ between } b \text{ and } x\ . \end{aligned}$$

We get it due to equality $(*)$ where Lagrange's mean value theorem is used for the function $f_m - f_n$ and the interval with endpoints b

and x. By Condition 2, for any given ε there exists an n_0 such that for every $m, n \geq n_0$ and every $c \in I$ one has that $|V| < \varepsilon$. Thus the sequence $(g_n(x))$ is uniformly Cauchy on $I \setminus \{b\}$ and the proof is complete. $\square$

• *Every continuous function has a primitive function.* To prove it we need one more tool.

Definition 14 (uniform continuity) *Let $M \subset \mathbb{R}$. The function $f\colon M \to \mathbb{R}$ is uniformly continuous (on M) if*

$$\forall \varepsilon\, \exists \delta\colon a \in M \Rightarrow f[U(a,\, \delta) \cap M] \subset U(f(a),\, \varepsilon)\ .$$

So one δ works for all points $a \in M$.

Theorem 15 (continuity and compactness) *Let $M \subset \mathbb{R}$ be a compact set. If a function $f\colon M \to \mathbb{R}$ is continuous then it is uniformly continuous.*

Proof. We suppose that $M \subset \mathbb{R}$ is compact and that $f\colon M \to \mathbb{R}$ is not uniformly continuous. So there is an $\varepsilon > 0$ such that for every n there are two points $a_n, b_n \in M$ such that $|a_n - b_n| < 1/n$ but $|f(a_n) - f(b_n)| \geq \varepsilon$. We use compactness of M and select from (a_n) and (b_n) convergent subsequences with limits in M. For simplicity of notation we assume that both (a_n) and (b_n) already converge and have limits $\lim a_n =: a \in M$ and $\lim b_n =: b \in M$. From $|a_n - b_n| < 1/n$ it follows that $a = b$. But from $|f(a_n) - f(b_n)| \geq \varepsilon$ and the convergence of (a_n) and (b_n) to a it follows that for every δ,

$$f[U(a,\, \delta) \cap M] \not\subset U(f(a),\, \varepsilon/2)\ .$$

Thus the function f is not continuous at a and f is not continuous on M. $\square$

Theorem 16 ($\exists$ antiderivative) *Suppose that $f\colon I \to \mathbb{R}$ is a continuous function defined on a nontrivial interval $I \subset \mathbb{R}$. Then f has a primitive function $F\colon I \to \mathbb{R}$.*

Brief proof. We first assume that I is compact, $I = [a, b]$ with $a < b$. A function $g\colon I \to \mathbb{R}$ is a *broken line* if it is continuous and there exists a partition $a = a_0 < a_1 < \cdots < a_k = b$ of I such that each restriction $g \,|\, [a_{i-1}, a_i]$ is linear, i.e., of the form $g(x) = c_i x + d_i$. By Theorem 15,

$$\forall\, n\ \exists\, \text{broken line } g_n\colon x \in I \Rightarrow |f(x) - g_n(x)| < 1/n\ .$$

Since $\int (cx + d) = cx^2/2 + dx + e$, according to Proposition 6 in the last lecture there exist $G_n\colon I \to \mathbb{R}$ such that $G_n = \int g_n$ and $G_n(a) = 0$. But then, since $g_n \rightrightarrows f$ (on I) and $G_n' = g_n$ on I, by Theorem 13 there exists an $F\colon I \to \mathbb{R}$ such that $G_n \to F$ (on I) and, especially, $F' = f$ on I, that is, $F = \int f$.

If the interval I is not compact, we write it as a union of nested non-trivial compact intervals I_n: $I_1 \subset I_2 \subset \dots$ and $\bigcup_{n\ge 1} I_n = I$. On each I_n we take an appropriate $F_n = \int f \,|\, I_n$ and then $F := \bigcup_{n\ge 1} F_n$ is a primitive function of f on I. $\square$

See `https://kam.mff.cuni.cz/~klazar/proofdet.pdf` for details of the proof (when I write them down). P. Lundström, Primitives of continuous functions via polynomials, `https://arxiv.org/abs/2204.05012` gives a similar proof, but with polynomials in place of broken lines. In a simpler way we prove the theorem later again by the Riemann integral.

THANK YOU FOR YOUR ATTENTION!