LECTURE 7, 3/30/2022[1]

DERIVATIVES OF FUNCTIONS

• *Derivatives of functions.* They are another fundamental notion of mathematical analysis. We encounter them far beyond the borders of analysis, especially in physics, but also in economical, biological, sociological and other models. Often derivative at a point is considered only for interior points. But then one cannot compute the derivative at 0 of the function, for example,

$$d(x) := \frac{\sin(1/x)}{\sin(1/x)} \ \ldots\ x \neq 0,\ d(0) := 1\ ,$$

with the definition domain $M = \mathbb{R} \setminus \{1/\pi n \mid n \in \mathbb{Z} \setminus \{0\}\} \ni 0$ that contains no neighborhood $U(0, \delta)$. And there are problems with differentiating inverse functions, as we explain later. Thus we take a more general road.

Definition 1 (derivatives of functions) *Let $a \in M$ be a limit point of the set $M \subset \mathbb{R}$ and $f = f(x)\colon M \to \mathbb{R}$ be a function. We set*

$$f'(a) = \frac{df}{dx}(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \stackrel{(*)}{=} \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

and say that the limit $f'(a) = \frac{df}{dx}(a) \in \mathbb{R}^*$ is the derivative of the function f at the point a.

The equality $(*)$ follows by two applications of the theorem on limits of composite functions (Theorem 14 in Lecture 5) or by a direct argument. For the finite derivative, i.e., if $f'(a) \in \mathbb{R}$, we say that

[1]Updated and corrected on April 15, 2022.

f is *differentiable* at a. Then we have for $x \in M$ that

$$f(x) = \underbrace{f(a) + f'(a) \cdot (x - a)}_{\text{linear approximation of } f} + \underbrace{o((x - a))}_{\text{its error}} \quad (x \to a)\ .$$

So near a the function f is closely approximated by the above linear function. We define one-sided derivatives.

Definition 2 (one-sided derivatives) *Suppose that $a \in M$ is a left, resp. right, limit point of $M \subset \mathbb{R}$, and that $f = f(x)\colon M \to \mathbb{R}$ is a function. We set*

$$f'_-(a) := \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} \overset{(*)}{=} \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}\ ,$$

resp.

$$f'_+(a) := \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \overset{(*)}{=} \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}\ ,$$

and say that $f'_-(a) \in \mathbb{R}^*$, resp. $f'_+(a) \in \mathbb{R}^*$, is the left-sided, resp. right-sided, derivative of f at the point a.

The two equalities $(*)$ are justified as for ordinary derivatives.

Derivatives and one-sided derivatives relate as follows. If f has the derivative $f'(a) \in \mathbb{R}^*$ then f has at least one one-sided derivative and $f'_-(a) = f'_+(a) = f'(a)$ whenever these values are defined. If the one-sided derivatives coincide, $f'_-(a) = f'_+(a) = L \in \mathbb{R}^*$, then also $f'(a) = L$. If $f'_-(a) \neq f'_+(a)$ then $f'(a)$ does not exist.

• *Derivatives and extremes.* Consider the function $f\colon [0, 1] \to \mathbb{R}$, $f(x) = x$, and the limit points 0 and 1 of its definition domain $[0, 1]$. Then

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x}{x} = 1$$

and, similarly, $f'(1) = 1$. At the same time f has at 0 a global minimum and at 1 a global maximum. But this means that the following theorem <u>does not hold</u>.

> *If a function has nonzero derivative at a limit point of the definition domain then it does not have local extreme at the point.*

There exist lecture notes that successfully "prove" this theorem. In order that we get it correctly below in Theorem 4, we introduce limit points of a special kind.

Definition 3 (TLP) *A point $a \in M$ is a two-sided limit point, abbreviated* TLP, *of the set $M \subset \mathbb{R}$ if*

$$\forall\, \delta\colon P^-(a,\, \delta) \cap M \neq \emptyset \neq P^+(a,\, \delta) \cap M\ .$$

So the point a is flanked on both sides by other arbitrarily close points of the set M. Every TLP of M is a limit point of M, but not the other way around.

Theorem 4 (necessary condition for extremes) *We assume that $b \in M$ is a* TLP *of $M \subset \mathbb{R}$ and that $f\colon M \to \mathbb{R}$ is a function such that $f'(b) \in \mathbb{R}^*$ exists and is nonzero. Then*

$$\forall\, \delta\ \exists\, c,\, d \in U(b,\, \delta) \cap M\colon f(c) < f(b) < f(d)$$

— the function f has no local extreme at b, it has at b neither local minimum nor local maximum.

Proof. Let b, M and f be as stated and let a δ be given. We assume that $f'(b) < 0$, the case with $f'(b) > 0$ is similar. We take

ε small enough so that $U(f'(b), \varepsilon) < \{0\}$ (i.e., $y \in U(f'(b), \varepsilon) \Rightarrow y < 0$). By Definition 1 there is a θ such that

$$x \in P(b,\, \theta) \cap M \Rightarrow \frac{\overbrace{f(x) - f(b)}^{\rightsquigarrow \text{ this is } < 0}}{x - b} \in U(f'(b),\, \varepsilon)\ .$$

Thus if $x \in P^-(b, \theta) \cap M$ then $f(x) > f(b)$, because $x - b < 0$ and the fraction is negative, and similarly if $x \in P^+(b, \theta) \cap M$ then $f(x) < f(b)$. We may assume that $\theta \le \delta$ and take any

$$c \in P^+(b,\, \theta) \cap M \ \text{ and } \ d \in P^-(b,\, \theta) \cap M\ .$$

The elements c and d exist due to the fact that b is a TLP of M. (Here the above mentioned lecture notes err if b is not a TLP of M.) Hence $c, d \in U(b, \delta) \cap M$, $f(c) < f(b)$ and $f(d) > f(b)$. $\square$

In another words, a function may have local extremes only in the points that (i) are not TLPs of the definition domain or (ii) are such that the derivative does not exist in them or (iii) are such that the derivative vanishes (i.e., is 0) in them.

• *Derivatives and continuity.* Differentiability of a function at a point is stronger property than its continuity at the point.

Proposition 5 (derivatives and continuity) *Let $b \in M$ be a limit point of $M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$ be a function. If $f'(b) \in \mathbb{R}$ (i.e., the derivative is finite) then f is continuous at b. The same holds for both one-sided derivatives and the corresponding one-sided continuity.*

Proof. By the theorem on arithmetic of limits of functions,

$$\begin{aligned}\lim_{x\to b} f(x) &= \lim_{x\to b}\left(f(b) + (x-b)\cdot\frac{f(x)-f(b)}{x-b}\right)\\ &= \lim_{x\to b} f(b) + \lim_{x\to b}(x-b)\cdot \lim_{x\to b}\frac{f(x)-f(b)}{x-b}\\ &= f(b) + 0\cdot f'(b)\\ &= f(b)\end{aligned}$$

and by Proposition 5 (Lect. 5) the function f is continuous at b. The same computation works for each one sided derivative, limit and continuity. $\square$

We apply the one-sided versions in the next lecture to convex and concave functions.

Clearly,

$$\mathrm{sgn}'(0) = \lim_{x\to 0}\frac{\mathrm{sgn}(x)-\mathrm{sgn}(0)}{x-0} = \frac{1}{0^+}, \frac{-1}{0^-} = +\infty\ .$$

Thus existence of an infinite derivative does not imply continuity at the point because $\mathrm{sgn}(x)$ is discontinuous at 0, it is even neither left-continuous nor right-continuous there.

In the second example we compute one-sided derivatives at 0 of $|\cdot|$. They are not equal,

$$(|x|)'_-(0) = \lim_{x\to 0^-}\frac{-x-0}{x-0} = -1 \;\text{ and }\; (|x|)'_+(0) = \lim_{x\to 0^+}\frac{x-0}{x-0} = 1\ ,$$

and $(|x|)'(0)$ does not exist. But $|x|$ is continuous at 0. Thus, of course, continuity at a point does not in general imply existence of a derivative.

In the third example we compute derivative of the square root function

$$\sqrt{x}\colon [0,\, +\infty) \to [0,\, +\infty)\ .$$

Let $a > 0$. Then

$$\begin{aligned}(\sqrt{x})'(a) \;&=\; \lim_{x\to a}\frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a}\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}\\ &=\; \lim_{x\to a}\frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{a}}\ .\end{aligned}$$

The last equality holds by the arithmetic of limits of functions. At 0 one has that

$$(\sqrt{x})'(0) = \lim_{x\to 0}\frac{\sqrt{x}-\sqrt{0}}{x-0} = \lim_{x\to 0}\frac{1}{\sqrt{x}} = \frac{1}{0} = +\infty\ .$$

But $\sqrt{x}$ is continuous at 0. Thus infinite derivative is compatible with continuity at the point. We could also write $(\sqrt{x})'_+(0)$ and it is not wrong, but it is misleading, because in terms of Definition 1 the left end 0 of the interval $[0,+\infty)$ is simply its limit point like any other. The notation $(\sqrt{x})'_+(0)$ must be used by those who define ordinary (two-sided) derivative only in interior points, because then they do not have it available in extreme points. The value $(\sqrt{x})'_-(0)$ is not defined because 0 is not the left limit point of $[0,+\infty)$.

In the last fourth example we compute derivatives of constants and of powers with natural exponents.

Proposition 6 **(c' a $(x^n)'$)** *The following formulas hold.*

1. *If for $c \in \mathbb{R}$ we denote by $f_c\colon \mathbb{R} \to \{c\}$ the constant function with the value c, then for every $a \in \mathbb{R}$ one has that*
$$f_c'(a) = 0\ .$$

2. *For every $n \in \mathbb{N}$ and every $a \in \mathbb{R}$,*
$$(x^n)'(a) = na^{n-1}\ .$$

Proof. 1. Let $a, c \in \mathbb{R}$. Then
$$f_c'(a) = \lim_{x\to a} \frac{f_c(x) - f_c(a)}{x-a} = \lim_{x\to a} \frac{c-c}{x-a} = \lim_{x\to a} 0 = 0\ .$$

2. Let $n \in \mathbb{N}$ and $a \in \mathbb{R}$. Then
$$\begin{aligned}(x^n)'(a) &= \lim_{x\to a} \frac{x^n - a^n}{x-a} \\ &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \cdots + a^{n-1})}{x-a} \\ &= \lim_{x\to a} (x^{n-1} + x^{n-2}a + \cdots + a^{n-1}) \\ &= \underbrace{a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n \text{ summands}} = na^{n-1}\ .\end{aligned}$$

The penultimate equality holds due to the arithmetic of limits of functions. $\square$

In Proposition 18 we show an example of a discontinuous derivative.

• *Geometry of derivatives: tangent lines.* This passage presents two (actually three) definitions of tangents, two possible approaches to this notion. For $M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$, the *graph* of the function f is the plane set

$$G_f := \{(x,\, f(x)) \mid x \in M\} \subset \mathbb{R}^2\ .$$

Here is the first definition of tangents.

Definition 7 (standard one) *Let $a \in M \subset \mathbb{R}$, where a is a limit point of M, and $f\colon M \to \mathbb{R}$ be a function that is differentiable at a. The tangent (line) to the graph of f at the point $(a, f(a)) \in G_f$ is the line ℓ given by the equation*

$$\ell\colon y = f'(a) \cdot (x - a) + f(a)\ .$$

Thus it is the only line with the slope $f'(a)$ going through the point $(a, f(a))$.

At this equation we could stop, and often texts on tangents stop here. However, we will go on and will show how to define tangents without derivatives. We read often that the tangent at a point B is somehow the limit of secants — lines going through two points of the graph — when their determining points go in limit to B. But details of this limit transition are never revealed. We shall present them here.

Let $(a, b), (a', b') \in \mathbb{R}^2$ be two distinct points in the plane. We define the *line* going through them as the set

$$\kappa(a,\, b,\, a',\, b') := \{(a,\, b) + t \cdot (a' - a,\, b' - b) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2\ . \quad (\kappa)$$

If $\lambda \subset \mathbb{R}^2$ is a line and $(a, b), (a', b') \in \mathbb{R}^2$ are two distinct points, then it is not hard to see that

$$\lambda = \kappa(a,\, b,\, a',\, b') \iff (a,\, b) \in \lambda \wedge (a', b') \in \lambda\ .$$

For a line, in its representation (κ) either always $a = a'$ or always $a \neq a'$. In the former case we speak of the *vertical* line and in the latter case of the *non-vertical* line. Let $M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$. A *secant* of the graph G_f of f is any line

$$\kappa(x,\ f(x),\ x',\ f(x')),\ x,\ x' \in M,\ x \neq x'\ .$$

Every secant is non-vertical. For a distinguished point $(a, f(a)) \in G_f$, the *main* secants go through it and through another *secondary* point of G_f. Other secants of G_f are *non-main*.

To any pair $(s, b) \in \mathbb{R}^2$ we can associate the set (which is in fact a line)

$$\ell(s,\ b) := \{(x,\ sx + b) \mid x \in \mathbb{R}\} \subset \mathbb{R}^2 \tag{ℓ}$$

with the *slope* s. It is not hard to see that (i) every set $\ell(s, b)$ is a non-vertical line, that (ii) every non-vertical line is of the form (ℓ) and that (iii) for every non-vertical line κ there is exactly one pair $(s, b) \in \mathbb{R}^2$ such that $\kappa = \ell(s, b)$. The map

$$(s,\ b) \mapsto \ell(s,\ b),\ \text{see } (\ell)\ ,$$

is therefore a bijection from $\mathbb{R}^2$ to the set of all non-vertical lines. Thus slopes of non-vertical lines are uniquely determined. The slope s of a non-vertical line $\kappa(a,\ b,\ a',\ b')$ is easy to compute, it is

$$s = \frac{b' - b}{a' - a}\ .$$

We define another kind of limit.

Definition 8 (limits of lines) *If ℓ is a non-vertical line, (ℓ_n) is a sequence of non-vertical lines and their (ℓ) representations $\ell = \ell(s, b)$ and $\ell_n = \ell(s_n, b_n)$ satisfy that*

$$\lim s_n = s \wedge \lim b_n = b \ ,$$

we write that $\lim \ell_n = \ell$ and say that the lines ℓ_n have the limit ℓ.

Limits of lines are unique because the (ℓ) representations are unique and so are the limits of real sequences. Here is the second definition of tangents.

Definition 9 (by limits) *Let $a \in M \subset \mathbb{R}$, where a is a limit point of M, $f\colon M \to \mathbb{R}$ be a function and let ℓ be a non-vertical line. If for every sequence $(x_n) \subset M \setminus \{a\}$ with $\lim x_n = a$ we have in the sense of Definition 8 that*

$$\lim \kappa(a,\ f(a),\ x_n,\ f(x_n)) = \ell \ ,$$

we say that the line ℓ is the tangent (line) to the graph of f at the point $(a, f(a)) \in G_f$.

Thus the tangent at $((a, f(a))$ is in this definition the limit of any sequence of main secants of the graph such that the secondary points go in limit to $(a, f(a))$. It is a (rigorous!) definition of tangents avoiding mentioning $f'(a)$. Below, in Theorem 11, we present the third definition of tangents that even does not mention the point $(a, f(a))$.

We show that the tangent ℓ at $(a, f(a))$ goes through this point.

We set

$$\kappa_n := \kappa(a, f(a), x_n, f(x_n)) = \ell(s_n, f(a) - s_n a)\ .$$

Then $\lim s_n = s$, where s is the slope of the tangent ℓ, and $\lim \kappa_n = \ell(s, f(a) - sa)$. By uniqueness of limits of lines, $\ell = \ell(s, f(a) - sa)$ and by (ℓ) one has that $(a, f(a)) \in \ell$.

The next theorem proves equivalence of both definitions.

Theorem 10 (equivalence of definitions) *Let $a \in M$, where a is a limit point of $M \subset \mathbb{R}$, $f\colon M \to \mathbb{R}$ be a function and ℓ be a non-vertical line. The next two claims are equivalent.*

1. *The line ℓ is tangent to G_f at $(a, f(a))$ by Definition 9.*
2. *The function f has the derivative $f'(a) \in \mathbb{R}$ and*

$$\ell = \ell(f'(a),\ f(a) - a \cdot f'(a))\ ,$$

so that ℓ is tangent to G_f at $(a, f(a))$ by Definition 7.

Proof. Let a, M, f and ℓ be as stated.

The implication $1 \Rightarrow 2$. We assume that ℓ is tangent to G_f at $(a, f(a))$ by Definition 9. Let $(x_n) \subset M \setminus \{a\}$ be any sequence with $\lim x_n = a$, let $\kappa_n := \kappa(a, f(a), x_n, f(x_n))$ and let s_n be the slope of the secant κ_n. By the assumption, $\lim \kappa_n = \ell$ and the formula for slopes gives that

$$\lim \frac{f(x_n) - f(a)}{x_n - a} = \lim s_n = s\ ,$$

where s is the slope of the line ℓ. By Heine's definition of limits of functions and Definition 1, $f'(a) = s$. We proved above that the tangent ℓ goes through $(a, f(a))$, hence

$$\ell = \ell(s,\ f(a) - a \cdot s) = \ell(f'(a),\ f(a) - a \cdot f'(a)) \ .$$

The implication $1 \Leftarrow 2$. We assume that the derivative $f'(a) \in \mathbb{R}$ exists and that the line ℓ is given by the stated formula. Let $(x_n) \subset M \setminus \{a\}$ be any sequence with $\lim\ x_n = a$. By the assumption and Heine's definition of limits of functions,

$$\lim \underbrace{\frac{f(x_n) - f(a)}{x_n - a}}_{s_n} = f'(a) \ .$$

Since the fraction s_n is the slope of the main secant

$$\kappa_n := \kappa(a, f(a), x_n, f(x_n)) = \ell(s_n, f(a) - s_n a) \ ,$$

these secants have the limit

$$\lim\ \kappa_n = \ell(f'(a),\ f(a) - f'(a) \cdot a) = \ell \ .$$

By Definition 9 the line ℓ is tangent to G_f at $(a, f(a))$. $\square$

This theorem is not so surprising. The next theorem is more interesting; unfortunately we have no time for its proof (later on see "*New look* ..." at `https://arxiv.org/`).

Theorem 11 (limits of non-main secants) *Let $a \in M$, a be a* TLP *of $M \subset \mathbb{R}$, $f\colon M \setminus \{a\} \to \mathbb{R}$ be a function and ℓ be a non-vertical line. The following two claims are equivalent.*

1. *The function f can be extended by the value $f(a)$ to $f\colon M \to \mathbb{R}$ so that ℓ is tangent to G_f at $(a, f(a))$ by Definition 9.*

2. *For every two sequences $(x_n), (x'_n) \subset M \setminus \{a\}$ satisfying that $\lim x_n = \lim x'_n = a$ and $x_n < a < x'_n$ for every n, we have by Definition 8 the limit of lines*

$$\lim \kappa(x_n,\, f(x_n),\, x'_n,\, f(x'_n)) = \ell\ .$$

So, according to the second part, the tangent is the limit of all those sequences of main secants of the graph, in which the pairs of determining points go in limit to $(a, f_0(a))$ and the points in each pairs are separated by $(a, f_0(a))$. Thus we give here a definition of tangent in the non-existent point $(a, f(a))$ of the graph G_f!

• *Arithmetic of derivatives.* We describe relations between derivatives and arithmetic operations.

Proposition 12 (linearity of derivatives) *Let $a \in M$, where a is a limit point of $M \subset \mathbb{R}$, $f, g \colon M \to \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then the equality*

$$(\alpha f(x))'(a) = \alpha f'(a)$$

holds whenever one side is defined, and the equality

$$(f(x) + g(x))'(a) = f'(a) + g'(a)$$

holds whenever the right-hand side is defined. The same formulas hold for one-sided derivatives.

Proof. By the arithmetic of limits of functions,

$$(\alpha f(x))'(a) = \lim_{x \to a} \frac{\alpha f(x) - \alpha f(a)}{x - a} = \alpha \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \alpha f'(a)\ .$$

Let $h(x) := f(x) + g(x)$. Then by the same theorem also

$$\begin{aligned} h'(a) &= \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\ &= f'(a) + g'(a) \end{aligned}$$

whenever the last expression is defined in the arithmetic of $\mathbb{R}^*$. For one-sided derivatives both computations work without change. □

For example (the definition domain is $[0, +\infty)$),

$$(\mathrm{sgn}(x) + \sqrt{x})'(0) = \mathrm{sgn}'(0) + (\sqrt{x})'(0) = +\infty + (+\infty) = +\infty\ .$$

As an exercise, compute $(\mathrm{sgn}(x) - \sqrt{x})'_+(0)$.

Theorem 13 (Leibniz formula) *Let $a \in M$, where a is a limit point of $M \subset \mathbb{R}$, and $f, g \colon M \to \mathbb{R}$ be functions. If f or g is continuous at a then*

$$(fg)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$$

if the right-hand side is defined. The same formula holds for one-sided derivatives and one-sided continuity.

Proof. Let g be continuous at a, the other case with f is symmetric. By the assumption and by the arithmetic of limits of functions,

$$\begin{aligned}
&(fg)'(a) = \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))g(x) + f(a)(g(x) - g(a))}{x-a} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \lim_{x\to a} g(x) + f(a) \lim_{x\to a} \frac{g(x) - g(a)}{x-a} \\
&= f'(a)g(a) + f(a)g'(a)\ .
\end{aligned}$$

For one-sided derivatives this computation does not change. $\square$

The formula bears the name of the German philosopher, mathematician and polymath *Gottfried W. Leibniz (1646–1716)*. Together with I. Newton Leibniz is considered to be the discoverer of the differential and integral calculus. The following example shows that for f and g discontinuous at a the Leibniz formula need not hold. This can occur only for infinite derivatives $f'(a)$ and $g'(a)$, finite derivatives at a imply continuity at a (Proposition 5).

Let $a := 0$ and the functions $f, g \colon \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = -g(x) := \operatorname{sgn} x \ \text{ for } \ x \neq 0,\ f(0) := -\tfrac{1}{2} \ \text{ and } \ g(0) := \tfrac{1}{2}\ .$$

Then the right-hand side of the Leibniz formula is $(+\infty)+(+\infty) = +\infty$, but the left-hand side is not defined.

Proposition 14 (derivatives of ratios) *Let $a \in M \subset \mathbb{R}$, where a is a limit point of M, and $f, g\colon M \to \mathbb{R}$ be functions. If $g(a) \neq 0$ and g is continuous at a then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{g(a)^2}\,,$$

if the right-hand side is defined. The same formula holds for one-sided derivatives and one-sided continuity.

Proof. The assumptions and arithmetic of limits of functions yield that

$$(f/g)'(a) = \lim_{x\to a} \frac{f(x)/g(x) - f(a)/g(a)}{x-a} =$$
$$\lim_{x\to a} \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(x)}{g(x)g(a)(x-a)} =$$
$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} \lim_{x\to a} \frac{g(a)}{g(x)g(a)} - \lim_{x\to a} \frac{f(a)}{g(x)g(a)} \lim_{x\to a} \frac{g(x)-g(a)}{x-a}$$
$$= \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}\,.$$

For one-sided derivatives the computation is very similar. □

Again there is an example showing that the formula need not hold for g discontinuous at a.

• *Derivatives of composite functions and of inverse functions.* We present formulas for these derivatives. We do not mention versions for one-sided derivatives and have no time for proofs. However, we present the second theorem in a more general and better form than can be found elsewhere.

Theorem 15 (derivatives of composite functions) *Let $a \in M$, where a is a limit point of $M \subset \mathbb{R}$, $g\colon M \to N$ be continuous at a, with $g'(a) \in \mathbb{R}^*$ and such that $g(a) \in N$ is a limit point of $N \subset \mathbb{R}$, and let $f\colon N \to \mathbb{R}$ have the derivative $f'(g(a)) \in \mathbb{R}^*$. Then the composite function*

$$f(g)\colon M \to \mathbb{R}$$

has the derivative

$$(f(g))'(a) = f'(g(a)) \cdot g'(a)$$

whenever this product is defined, i.e., is neither $0 \cdot (\pm\infty)$ nor $(\pm\infty) \cdot 0$.

Proof. `https://kam.mff.cuni.cz/~klazar/compfun.pdf`, when I write it down. □

[2]Standard formulations of the theorem on derivatives of inverse functions are too restrictive, because of derivatives defined only in interior points. One such formulation is this.

> Let I be a non-degenerate interval and let a be an inner point of I. Let f be a continuous and strictly monotone function on I. Let $b = f(a)$. Then the following hold.
>
> (a) If f has a nonzero derivative at the point a, then
>
> $$(f^{-1})'(b) = \frac{1}{f'(a)}.$$

[2]This remark is intended for mathematical analysis teachers.

(b) If $f'(a) = 0$ and f is increasing on I, then

$$(f^{-1})'(b) = \infty.$$

(c) If $f'(a) = 0$ and f is decreasing on I, then

$$(f^{-1})'(b) = -\infty.$$

We see that in order for b to be an inner point of the definition domain of f^{-1} (so that $(f^{-1})'(b)$ has a chance to exist), one imposes on f additional restrictions and regulates its behavior *outside the point a*. The "correct" theorem on derivatives of inverse functions, which we now introduce, must not assume anything about the behavior of f outside of a (of course it has to assume injectivity of f) and of f^{-1} outside of b beyond the increase or decrease of f at a, existence of $f'(a)$ and continuity of f^{-1} at b. Such a theorem necessarily has to use more general derivatives, defined at any limit point of the definition domain.

We define that a function $f\colon M \to \mathbb{R}$ *increases*, resp. *decreases, at a point $a \in M \subset \mathbb{R}$* if for some δ one has that

$$x \in P^-(a,\, \delta) \cap M,\, x' \in P^+(a,\, \delta) \cap M \Rightarrow f(x) < f(a) < f(x')\ ,$$

resp. the opposite inequalities hold. So here is our version of the theorem on derivatives of inverse functions. It is of course for derivatives in the sense of Definition 1.

Theorem 16 (derivatives of inverse functions) *Let $a \in M$, a be a limit point of $M \subset \mathbb{R}$, $f\colon M \to \mathbb{R}$ be an injective function with the derivative $f'(a) \in \mathbb{R}^*$ and let the inverse function $f^{-1}\colon f[M] \to M$ be continuous at $b := f(a)$. Then the following hold.*

1. *If $f'(a) \in \mathbb{R} \setminus \{0\}$ then f^{-1} has the derivative*

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}\ .$$

2. *If $f'(a) = 0$ and f increases, resp. decreases, at the point a then f^{-1} has the derivative*

$$(f^{-1})'(b) = +\infty,\ \mathit{resp.}\ (f^{-1})'(b) = -\infty\ .$$

3. *If $f'(a) = \pm\infty$ and b is a limit point of $f[M]$ then f^{-1} has the derivative*

$$(f^{-1})'(b) = 0\ .$$

Proof. See `https://kam.mff.cuni.cz/~klazar/inverfun.pdf` when I write it down. □

• *A table of derivatives of elementary functions.* We present formulas for these derivatives. We already proved a couple of them.

Theorem 17 (a table of derivatives) *We have the following derivatives.*

1. *On* $\mathbb{R}$, $\exp(x)' = \exp(x)$, $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(\arctan x)' = 1/(1+x^2)$, $(\mathrm{arccot}\, x)' = -1/(1+x^2)$, $(x^n)' = nx^{n-1}$ *for* $n \in \mathbb{N}$ *and* $c' = 0$ *for every* $c \in \mathbb{R}$.
2. *On* $\mathbb{R} \setminus \{0\}$, $(x^b)' = bx^{b-1}$ *for every negative* $b \in \mathbb{Z}$.
3. *On* $(0, +\infty)$, $(x^b)' = bx^{b-1}$ *for every* $b \in \mathbb{R} \setminus \mathbb{Z}$ *and* $(\log x)' = 1/x$.
4. *On* $\mathbb{R} \setminus \{k\pi + \pi/2 \mid k \in \mathbb{Z}\}$, $(\tan x)' = 1/(\cos x)^2$.
5. *On* $\mathbb{R} \setminus \{k\pi \mid k \in \mathbb{Z}\}$, $(\cot x)' = -1/(\sin x)^2$.
6. *On* $(-1, 1)$, $(\arcsin x)' = 1/\sqrt{1-x^2}$ *and* $(\arccos x)' = -1/\sqrt{1-x^2}$.

Proof. See `https://kam.mff.cuni.cz/~klazar/tablederi.pdf` when I write it down. $\square$

Sometimes it is incorrectly claimed that the derivative of a function has to be continuous. This probably arises by confusion with the correct proposition that a function is continuous at any point where it is differentiable. In the closing we therefore present an example of a function with discontinuous derivative.

Proposition 18 (discontinuous derivative) *The function $f\colon \mathbb{R} \to \mathbb{R}$, given by $f(x) := x^2 \sin(1/x)$ for $x \neq 0$ and else by $f(0) := 0$, has the everywhere defined derivative*

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & \dots \quad x \neq 0 \quad \textit{and} \\ 0 & \dots \quad x = 0\ , \end{cases}$$

which is discontinuous at 0.

Proof. For $x \neq 0$ the formula for $f'(x)$ follows by the Leibniz formula, the formula for derivative of sinus, the formula for derivative of a composite function and the formula for derivative of x^b. At 0 we have by Definition 1 that

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} x\sin(1/x) = 0\ .$$

It is clear that f' is discontinuous at 0 because for $x \to 0$ also $2x\sin(1/x) \to 0$ but the function $\cos(1/x)$ oscillates with frequency going to ∞ in the whole interval $[-1, 1]$. $\square$

THANK YOU FOR YOUR ATTENTION!