LECTURE 6, 3/23/2022
PROPERTIES OF CONTINUOUS FUNCTIONS

• *Heine's definition of continuity at a point.* From the last lecture we know that continuity of a function $f\colon M \to \mathbb{R}$ at a point $a \in M \subset \mathbb{R}$ means that

$$\forall\,\varepsilon\;\exists\,\delta:\;\; f[U(a,\,\delta) \cap M] \subset U(f(a),\,\varepsilon)\;.$$

In this lecture we will refer frequently (9×, to be precise) to the next result.

Proposition 1 (Heine's definition) *$f\colon M \to \mathbb{R}$ is continuous at a point $a \in M \subset \mathbb{R}$ if and only if*

$$\forall\,(a_n) \subset M:\; \lim a_n = a \Rightarrow \lim f(a_n) = f(a)\;.$$

Proof. We proved this equivalence as $1 \iff 3$ in Proposition 5 in the last lecture for limit points. If $a \in M$ is an isolated point of M then f is continuous at a by Proposition 7 in the last lecture. But then $\lim a_n = a$ means that $a_n = a$ for every $n \ge n_0$. Hence $f(a_n) = f(a)$ for every $n \ge n_0$ and $\lim f(a_n) = f(a)$. □

Definition 2 (continuity on a set) *Let $M \subset \mathbb{R}$ and let $f\colon M \to \mathbb{R}$. The function f is continuous (on M) if f is continuous at every point of M.*

• *Dense sets.* We introduce the relation of density of a set in another set.

Definition 3 (dense sets) *Let $N \subset M \subset \mathbb{R}$. We say that the set N is dense in the set M if*

$$\forall\, a \in M \ \forall\, \delta: \ U(a,\, \delta) \cap N \neq \emptyset\ .$$

Let $N \subset M \subset \mathbb{R}$. Clearly, N is dense in M iff for every point $a \in M$ there is a sequence $(b_n) \subset N$ such that $\lim\, b_n = a$. For example, the set of fractions $\mathbb{Q}$ is dense in $\mathbb{R}$.

Proposition 4 (density and continuity) *Suppose that $N \subset M \subset \mathbb{R}$, that N is dense in M and that $f,\, g\colon M \to \mathbb{R}$ are two continuous functions such that $\forall\, x \in N\colon f(x) = g(x)$. Then*

$$f = g$$

— the functions f and g coincide.

Proof. Let $y \in M$ be any point and $(a_n) \subset N$ be a sequence with $\lim\, a_n = y$. Then

$$f(y) = f\left(\lim\, a_n\right) = \lim\, f(a_n) = \lim\, g(a_n) = g\left(\lim\, a_n\right) = g(y)\ .$$

Here the 2nd and 4th equality follow from Proposition 1. The 3rd equality follows from the assumption that f and g are equal on N. Thus $f = g$ completely. $\square$

Recall that if $A \subset B$ and C are sets and $f\colon B \to C$ is a function, its *restriction* to A is the function $f \mid A\colon A \to C$ given by $\forall\, x \in A\colon (f \mid A)(x) := f(x)$.

Theorem 5 (H. Blumberg, 1922) *For any function $f\colon \mathbb{R} \to \mathbb{R}$ there is a set $M \subset \mathbb{R}$ dense in $\mathbb{R}$ and such that the restriction $f \,|\, M$ is a continuous function.*

Henry Blumberg (1886–1950) was an American mathematician who was born in Lithuania.

• *Counting continuous functions.* For $M \subset \mathbb{R}$ we introduce the notation

$$C(M) := \{f\colon M \to \mathbb{R} \mid f \text{ is continuous}\} \ .$$

It is the set of all continuous real functions defined on the set M. The next theorem is a basic result in set theory.

Theorem 6 (Cantor–Bernstein) *If there exist injections $f\colon X \to Y$ and $g\colon Y \to X$ then there is a bijection*

$$h\colon X \to Y \ .$$

The map h can be chosen so that for every $x \in X$ one has that $h(x) = f(x)$ or $h(x) = g^{-1}(x)$.

How many continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ are there? That many as the real numbers.

Theorem 7 (counting cont. functions) *There exists a bijection $h\colon \mathbb{R} \to C(\mathbb{R})$.*

Proof. By the previous theorem it suffices to find injections $f\colon \mathbb{R} \to C(\mathbb{R})$ and $g\colon C(\mathbb{R}) \to \mathbb{R}$. The former one is obvious,

$$f(a) := (b \mapsto a) \ ,$$

i.e., $f(a)$ is the constant function with the value a.

We describe the latter injection $g\colon C(\mathbb{R}) \to \mathbb{R}$. We view the numbers in $\mathbb{R}$ as infinite decimal expansions, for instance $-\pi = -3.141592\ldots$ or $2022.00000\ldots$. By Proposition 4 every function $j \in C(\mathbb{R})$ is completely determined by the countably many values $j(x)$, $x \in \mathbb{Q}$. Let $r\colon \mathbb{N} \to \mathbb{Q}$ and $s\colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be bijections, for example $(k, l, n \in \mathbb{N})$

$$s(n) = s(2^{k-1} \cdot (2l - 1)) = (s_1(n),\, s_2(n)) := (k,\, l)\ .$$

We encode the decimal digits 0, 1, ..., 9, the decimal point . and the minus sign $-$ by two decimal digits:

$$c(0) := 00,\ c(1) := 01,\ \ldots,\ c(9) := 09,\ c(.) := 10\ \text{ and }\ c(-) := 11\ .$$

The map $g\colon C(\mathbb{R}) \to \mathbb{R}$ has at the function $j \in C(\mathbb{R})$ the value

$$g(j) := 0.a_1\, a_2\, a_3\ \ldots\ a_{2n-1}\, a_{2n}\ \ldots\ =: \alpha\ .$$

The digits $a_n \in \{0, 1, \ldots, 9\}$ are defined as follows. For $k, l \in \mathbb{N}$ we consider the decimal expansions

$$j(r(k)) =: b(1,\, k)\, b(2,\, k)\ \ldots\ b(l,\, k)\ \ldots$$

of the values $j(r(k))$ of the function j on the fractions $r(k) \in \mathbb{Q}$, with symbols $b(l, k) \in \{0, 1, \ldots, 9,\ .,\, -\}$. Then we set

$$a_{2n-1}\, a_{2n} = c(b(l, k)) := c(b(s_1(n),\, s_2(n)))\ .$$

A short meditation reveals that the map g is injective: the single decimal expansion α stores all values of the function j on all rational numbers. $\square$

• *Attaining intermediate values by continuous functions.* The image of the function $\mathrm{sgn}(x)$ is $\{-1, 0, 1\}$, but nothing else between these three points. Images of intervals by continuous functions cannot look like this.

Theorem 8 (on intermediate values) *Let $a, b, c \in \mathbb{R}$, $a < b$, $f\colon [a, b] \to \mathbb{R}$ be a continuous function and let $f(a) < c < f(b)$ or $f(a) > c > f(b)$. Then*

$$\exists\, d \in (a, b)\colon\ f(d) = c\ .$$

Proof. We suppose that $f(a) < c < f(b)$, the case that $f(a) > c > f(b)$ is treated similarly. Let

$$A := \{x \in [a, b] \mid f(x) < c\} \ \text{ and } \ d := \sup(A) \in [a,\, b]\ .$$

The number d is correctly defined because the set A is nonempty ($a \in A$) and bounded from above (for instance, b is an upper bound). We show that both $f(d) < c$ and $f(d) > c$ lead to contradiction, so that $f(d) = c$. The continuity of f at a and b implies that $d \in (a, b)$. Let $f(d) < c$. The continuity of f at d implies that there is a δ such that $x \in U(d, \delta) \cap [a, b] \Rightarrow f(x) < c$. But then A contains numbers larger than d, in contradiction with the fact that d is an upper bound of A. Let $f(d) > c$. In the same vein, there is a δ such that $x \in U(d, \delta) \cap [a, b] \Rightarrow f(x) > c$. But then every $x \in [a, d)$ sufficiently close to d lies outside of A, in contradiction with the fact that d is the smallest upper bound of A. $\square$

Corollary 9 (cont. image of an interval) *Let $I \subset \mathbb{R}$ be an interval (i.e., a convex set) and $f\colon I \to \mathbb{R}$ be a continuous function. Then*

$$f[I] = \{f(x) \mid x \in I\} \subset \mathbb{R}$$

is an interval too.

Proof. Theorem 8 shows that the set $f[I]$ is convex. □

You may wish to attempt the following corollary of the theorem on intermediate values as an exercise.

Corollary 10 (on climbing) *A climber starts climbing a mountain at midnight and reaches the summit exactly after 24 hours, again at midnight. Then the climber descends, again for exactly 24 hours, in the base camp. Prove that there is a time $t_0 \in [0, 24]$ when the climber is in both days in the same altitude.*

We prove the next corollary. Recall that a function $f\colon M \to \mathbb{R}$ is *increasing*, resp. *decreasing* (on $M \subset \mathbb{R}$), if for every $x, y \in M$ one has that $x < y \Rightarrow f(x) < f(y)$, resp. $f(x) > f(y)$.

Corollary 11 (continuity and inject. on an interval) *Suppose that $I \subset \mathbb{R}$ is an interval and that $f\colon I \to \mathbb{R}$ is a continuous injective function. Then f is either increasing or decreasing.*

Proof. If f neither increases nor decreases then there exist three number $a < b < c$ in I such that $f(a) < f(b) > f(c)$ or $f(a) >$

$f(b) < f(c)$. In the former case every d satisfying $f(a), f(c) < d < f(b)$ is attained, by Theorem 8, as $d = f(x) = f(y)$ for some $x \in (a, b)$ and $y \in (b, c)$. This contradicts the injectivity of f. In the latter case we get a very similar contradiction. $\square$

• *Continuous functions on compact sets.* Compact sets play in analysis and elsewhere (e.g., in optimization) an important role.

Definition 12 (compact sets) *A set $M \subset \mathbb{R}$ is compact if every sequence $(a_n) \subset M$ has a convergent subsequence (a_{m_n}) with $\lim a_{m_n} \in M$.*

By the Bolzano–Weierstrass theorem and the theorem on limits of sequences and order we know that every interval $[a, b]$ is compact. We characterize compact sets later and now prove on them an important theorem.

Theorem 13 (the min-max principle) *Let $M \subset \mathbb{R}$ be a nonempty compact set and $f\colon M \to \mathbb{R}$ be a continuous function. Then there exist points $a, b \in M$ such that*

$$\forall\, x \in M\colon f(a) \le f(x) \le f(b)\ .$$

We say that f attains at $a \in M$ its minimum (smallest value) $f(a)$ on M and that f attains at $b \in M$ its maximum (largest value) $f(b)$ on M.

Proof. We only prove the existence of the maximum of f, the proof for the minimum is very similar. Clearly, $f[M] \neq \emptyset$ and we show that this set is bounded from above. Suppose not, then there is a sequence $(a_n) \subset M$ such that $\lim f(a_n) = +\infty$. By the compactness of M the sequence (a_n) has a convergent subsequence (a_{m_n}) with $a := \lim\, a_{m_n} \in M$. Then $\lim\, f(a_{m_n}) = +\infty$ too. But this contradicts the fact that by Proposition 1, $\lim\, f(a_{m_n}) = f(a)$. Thus we can define

$$s := \sup(f[M]) \in \mathbb{R}$$

and by the definition of supremum there is a sequence $(a_n) \subset M$ with $\lim\, f(a_n) = s$. Due to compactness of M the sequence (a_n) has a convergent subsequence (a_{m_n}) with $b := \lim\, a_{m_n} \in M$. By Proposition 1 one has that $\lim\, f(a_{m_n}) = f(b) = s$. Since $s = f(b)$ is an upper bound of $f[M]$, we have that $f(b) \geq f(x)$ for every $x \in M$. $\square$

For non-compact M the theorem need not hold. For example, the function $f\colon [0,1) \to \mathbb{R}$, $f(x) = \frac{1}{1-x}$, is continuous but not bounded from above and does not have maximum. The function $f\colon [0,1) \to \mathbb{R}$, $f(x) = x$, is continuous and bounded from above but still does not have maximum. We review the standard classification of minima and maxima of functions as “global” and “local”.

Definition 14 (global and local) *Let $a \in M \subset \mathbb{R}$ and let $f\colon M \to \mathbb{R}$ be any function. The function f has on M a global maximum, resp. a global minimum, at a if*

$$\forall\, x \in M\colon f(x) \le f(a), \quad \textit{resp.} \quad f(x) \ge f(a)\ .$$

The function f has on M a local maximum, resp. a local minimum, at a if

$$\exists\, \delta\ \forall\, x \in U(a,\, \delta) \cap M\colon f(x) \le f(a), \quad \textit{resp.} \quad f(x) \ge f(a)\ .$$

When strict inequalities ($<$, resp. $>$) hold for every $x \ne a$, we speak of a strict global maximum, etc.

• *Compact sets in $\mathbb{R}$.* We know when a set $M \subset \mathbb{R}$ is bounded: $\exists\, c\ \forall\, a \in M\colon |a| < c$. It is *closed* if

$$\forall\, (a_n) \subset M\colon \lim a_n = a \Rightarrow a \in M\ .$$

It is *open* if

$$\forall\, a \in M\ \exists\, \delta:\ U(a,\, \delta) \subset M\ .$$

Proposition 15 (closed sets) *A set $M \subset \mathbb{R}$ is closed if and only if the set $\mathbb{R} \setminus M$ is open.*

Proof. $\mathbb{R} \setminus M$ is not open iff there is a point $a \in \mathbb{R} \setminus M$ such that for every δ, $U(a, \delta) \cap M \ne \emptyset$. Equivalently (choosing for every n some $a_n \in U(a, 1/n) \cap M$), there is a point $a \in \mathbb{R} \setminus M$ and a sequence $(a_n) \subset M$ such that $\lim a_n = a$. Equivalently, M is not closed. $\square$

Using the following structural description of open sets one can relatively easily imagine them. By *open intervals* we mean in it the intervals $(-\infty, a)$, $(a, +\infty)$ and (a, b) for $a < b$.

Proposition 16 (structure of open sets) *A set $M \subset \mathbb{R}$ is open if and only if there is a system of open intervals $\{I_j \mid j \in X\}$ such that the index set X is at most countable, the intervals I_j are mutually disjoint and*

$$\bigcup_{j \in X} I_j = M \ .$$

Closed sets are complements of open sets and therefore they are unions of "gaps" between the above intervals I_j. If $|X| = n \in \mathbb{N}_0$, there are at most $n + 1$ gaps. What is hard to imagine is that for countable X the set of gaps may be uncountable. This is the reason that it is harder to imagine closed sets.

Theorem 17 (compact sets) *Let $M \subset \mathbb{R}$. Then M is compact if and only if M is closed and bounded.*

Proof. Let $M \subset \mathbb{R}$ be closed and bounded and let $(a_n) \subset M$ be any sequence. Since (a_n) is bounded, by the Bolzano–Weierstrass theorem it has a convergent subsequence (a_{m_n}) with $a := \lim a_{m_n} \in \mathbb{R}$. Since M is closed, $a \in M$. Thus M is compact.

Suppose that $M \subset \mathbb{R}$ is not bounded. We construct a sequence $(a_n) \subset M$ such that $|a_m - a_n| > 1$ for every two indices $m \neq n$. This property is inherited by every subsequence which therefore cannot be convergent and M is not compact. The first term $a_1 \in M$ is taken arbitrarily. Suppose that $a_1, a_2, \ldots, a_n$ have been defined such that $|a_i - a_j| > 1$ for every i, j with $1 \leq i < j \leq n$. Since M is not bounded, there is a point $a_{n+1} \in M$ such that $|a_{n+1}| > 1 +$

$\max(|a_1|, \ldots, |a_n|)$. Then $|a_{n+1} - a_i| > 1$ for every $i = 1, 2, \ldots, n$. In this way we define the whole (a_n).

Suppose that $M \subset \mathbb{R}$ is not closed. Then there is a convergent sequence $(a_n) \subset M$ such that $a := \lim a_{m_n} \in \mathbb{R} \setminus M$. Every subsequence has the same limit a, and so it does not have limit in M. Thus M is not compact. $\square$

• *Continuity and various operations.* We present several operations which produce new continuous functions from old ones. Recall that for two functions $f, g\colon M \to \mathbb{R}$ their *sum*, *product* and *ratio* function is defined as ($x \in M$)

$$\begin{aligned}(f+g)(x) &:= f(x) + g(x),\\ (fg)(x) &:= f(x) \cdot g(x) \text{ and}\\ (f/g)\,(x) &:= f(x)/g(x)\ ,\end{aligned}$$

respectively.

Proposition 18 (arithmetic of continuity) *Let $M \subset \mathbb{R}$ and $f, g\colon M \to \mathbb{R}$ be continuous functions. Then the sum and product function*

$$f + g,\ fg\colon M \to \mathbb{R}$$

are continuous. If $g \neq 0$ on M then also the ratio function

$$f/g\colon M \to \mathbb{R}$$

is continuous.

Proof. All three proofs are similar and we only prove the part with the ratio function. Let $a \in M$ be any point and $(a_n) \subset M$

be any sequence with $\lim a_n = a$. By Proposition 1 (implication $\Rightarrow$) one has that $\lim f(a_n) = f(a)$ and $\lim g(a_n) = g(a)$. By the theorem on arithmetic of limits of sequences,

$$\begin{aligned} \lim\,(f/g)(a_n) &= \lim f(a_n)/g(a_n) = \lim f(a_n)/\lim g(a_n) \\ &= f(a)/g(a) = (f/g)(a)\ . \end{aligned}$$

By Proposition 1 (implication $\Leftarrow$), the function f/g is continuous at the point a. □

Rational functions $r(x)$ are ratios of two polynomials, i.e., function of the form

$$r(x) := \frac{a_m x^m + \cdots + a_1 x + a_0}{b_n x^n + \cdots + b_1 x + b_0} \colon M \to \mathbb{R}\ ,$$

where $a_i, b_i \in \mathbb{R}$, $m, n \in \mathbb{N}_0$ and $a_m b_n \neq 0$; in the numerator we allow also the identically zero polynomial. The definition domain M of this function is the set

$$M = \mathbb{R} \setminus \{z_1,\, z_2,\, \ldots,\, z_k\}\ ,$$

where $z_i \in \mathbb{R}$ are all real roots of the polynomial in the denominator ($k \in \mathbb{N}_0$ and $k \leq n$).

Corollary 19 (continuity of rational functions)
Every rational function is continuous on its definition domain.

Proof. The identical function $f(x) = x$ and the constant functions $f(x) = c$, $c \in \mathbb{R}$, are continuous on $\mathbb{R}$. Starting with them and repeatedly applying the previous proposition we obtain that every rational function is continuous. □

All earlier mentioned elementary functions $\exp(x)$, $\log x$, $\cos x$, $\sin x$, a^x $(a \geq 0)$, $\arccos x$, $\arcsin x$, $\tan x$, $\arctan x$, $\cot x$ and $\operatorname{arccot} x$ are continuous on their definition domains.

Proposition 20 (continuity and composition) *Let $M, N \subset \mathbb{R}$ and let $g\colon M \to N$ and $f\colon N \to \mathbb{R}$ be continuous functions. Then the composite function*

$$f(g)\colon M \to \mathbb{R}$$

is continuous.

Proof. Let $a \in M$ be any point and $(a_n) \subset M$ be any sequence with $\lim a_n = a$. By Proposition 1 (implication $\Rightarrow$) one has that $\lim g(a_n) = g(a)$ and also that

$$\lim f(g)(a_n) = \lim f(g(a_n)) = f(g(a)) = f(g)(a)\ .$$

By Proposition 1 (implication $\Leftarrow$), $f(g)$ is continuous at a. $\square$

We know that every injection $f\colon A \to B$ has the inverse function (or inverse) $f^{-1}\colon f[A] \to A$ that is given by

$$\forall\, y \in f[A]\ \forall\, x \in A\colon f^{-1}(y) = x \iff f(x) = y\ .$$

Theorem 21 (continuity of inverses) *Let $M \subset \mathbb{R}$ and let $f\colon M \to \mathbb{R}$ be a continuous injective function. Then the inverse $f^{-1}\colon f[M] \to M$ is continuous if (i) M is compact or (ii) M is an interval.*

Proof. (i) We assume that M is compact, $b \in f[M]$ is any point and that $(b_n) \subset f[M]$ is any sequence with $\lim b_n = b$. We set $a := f^{-1}(b) \in M$ and $a_n := f^{-1}(b_n) \in M$. We show that

$\lim a_n = a$, which by Proposition 1 proves the continuity of f^{-1} at b. Let (a_{m_n}) be any subsequence of the sequence $(a_n) \subset M$ with $\lim a_{m_n} = L \in \mathbb{R}^*$. But $L \in M$ because M is bounded and closed (by Theorem 17). By Proposition 1, $\lim f(a_{m_n}) = f(L) = b$ because $(f(a_{m_n}))$ is a subsequence of the sequence (b_n). Due to the injectivity of f, $L = a$. Thus the sequence (a_n) does not have two subsequences with different limits and by part 2 of Proposition 6 in Lecture 2 (a_n) has a limit. We have just proven that this limit is a.

(ii) Let M be an interval. By Corollary 11 the function f increases or decreases. Suppose that f is decreasing, the increasing case is similar. By Corollary 9 the image $f[M]$ is an interval. Let $b \in f[M]$ and let an ε be given. We show that f^{-1} is right-continuous at b. This is trivial when b is the right endpoint of the interval $f[M]$ because then $U^+(b, \delta) \cap f[M] = \{b\}$. Suppose that b is not the right endpoint of this interval. Since f^{-1} is decreasing, $a := f^{-1}(b) \in M$ is not the left endpoint of the interval M and we can assume that ε is so small that $[a - \varepsilon, a] \subset M$. We set $\delta := f(a - \varepsilon) - f(a) = f(a - \varepsilon) - b$. Since f^{-1} decreases, it maps $[b, b + \delta] \subset f[M]$ to $[a - \varepsilon, a] \subset M$. Hence

$$f^{-1}[U^+(b,\, \delta) \cap f[M]] \subset U(f^{-1}(b),\, \varepsilon) = U(a,\, \varepsilon)$$

and f^{-1} is right-continuous at b. The left continuity is proven similarly and we see that f^{-1} is continuous at b. $\square$

The theorem also holds for (iii) open M and (iv) closed M if f increases or decreases, but we skip these proofs here. Part (ii) of the theorem implies that $\log x$ and inverse trigonometric functions are continuous.

THANK YOU FOR YOUR ATTENTION!