LECTURE 5, 3/16/2022

PROPERTIES OF LIMITS OF FUNCTIONS. CONTINUITY OF A FUNCTION AT A POINT

• *One-sided limits of functions.* In contrast with $\mathbb{C}$ or with the spaces $\mathbb{R}^n$ of dimension $n \geq 2$, deletion of one point disconnects the real axis in two separated pieces. So in $\mathbb{R}$ there are exactly two directions to approach in limit the given point, and hence the left-sided and right-sided limits. They only concern finite points, not infinities.

Definition 1 (one-sided neighborhoods) *For $\varepsilon, b \in \mathbb{R}$, the left, resp. right, ε-neighborhood of the point b is*

$$U^-(b,\, \varepsilon) := (b - \varepsilon,\, b],\ \textit{resp. }\ U^+(b,\, \varepsilon) := [b,\, b + \varepsilon)\ .$$

The left, resp. right, deleted ε-neighborhood of b is

$$P^-(b,\, \varepsilon) := (b - \varepsilon,\, b),\ \textit{resp. }\ P^+(b,\, \varepsilon) := (b,\, b + \varepsilon)\ .$$

So again $P^-(b,\, \varepsilon) = U^-(b,\, \varepsilon) \setminus \{b\}$ and $P^+(b,\, \varepsilon) = U^+(b,\, \varepsilon) \setminus \{b\}$. By means of these neighborhoods we define one-sided limit points.

Definition 2 (one-sided limit points) *A point $b \in \mathbb{R}$ is a left, resp. right, limit point of $M \subset \mathbb{R}$ if*

$$\forall\, \delta > 0 :\ \ P^-(b,\, \delta) \cap M \neq \emptyset\ ,$$

resp.

$$\forall\, \delta > 0 :\ \ P^+(b,\, \delta) \cap M \neq \emptyset\ .$$

As before b is a left (resp. right) limit point of M iff there is a sequence (a_n) in $(-\infty, b) \cap M$ (resp. in $(b, +\infty) \cap M$) such that

$\lim a_n = b$. A left (resp. right) limit point of a set is its limit point. Any limit point of a set is its left or right limit point, but it need not be its left *and* right limit point.

Definition 3 (one-sided limits) *Let $a \in \mathbb{R}$, $L \in \mathbb{R}^*$, $M \subset \mathbb{R}$, a be a left (resp. right) limit point of M and let*

$$f\colon M \to \mathbb{R}\ .$$

We write $\lim_{x\to a^-} f(x) = L$, resp. $\lim_{x\to a^+} f(x) = L$, and say that the function f has at the point a the left-sided, resp. right-sided, limit L if

$$\begin{aligned} \forall\, \varepsilon\ \exists\, \delta:\quad & f[P^-(a,\delta)\cap M] \subset U(L,\varepsilon)\ , \\ & \textit{resp. } f[P^+(a,\delta)\cap M] \subset U(L,\varepsilon)\ . \end{aligned}$$

It always holds that

$$\lim_{x\to a} f(x) = L \Rightarrow \lim_{x\to a^{\pm}} f(x) = L$$

or the one-sided limit of f at a is not defined because a is not the respective left or right limit point of the definition domain. It always holds that

$$\lim_{x\to a^-} f(x) = L \wedge \lim_{x\to a^+} f(x) = L \Rightarrow \lim_{x\to a} f(x) = L\ .$$

But it may be that $\lim_{x\to a^-} f(x) = L \neq L' = \lim_{x\to a^+} f(x)$. Then $\lim_{x\to a} f(x)$ does not exist. For instance, the function *signum*

$$\mathrm{sgn}(x)\colon \mathbb{R} \to \{-1,\,0,\,1\}\ ,$$

defined as $\mathrm{sgn}(x) = -1$ for $x < 0$, $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(x) = 1$ for

$x > 0$, has at 0 different one-sided limits

$$\lim_{x \to 0^-} \operatorname{sgn}(x) = -1 \quad \text{and} \quad \lim_{x \to 0^+} \operatorname{sgn}(x) = 1 \ .$$

Hence $\lim_{x \to 0} \operatorname{sgn}(x)$ does not exist. Like two-sided limits, one-sided limits are unique and have equivalent Heine definitions.

- *Continuity at a point.* The next definition is fundamental.

Definition 4 (continuity of a function at a point) *Let $a \in M \subset \mathbb{R}$ and let $f \colon M \to \mathbb{R}$. The function f is continuous at the point a if*

$$\forall\, \varepsilon\ \exists\, \delta:\ \ f[U(a,\, \delta) \cap M] \subset U(f(a),\, \varepsilon)\ .$$

Compared to the limit of f at a, the element L is replaced with the value $f(a)$, and $P(a, \delta)$ is replaced with the larger neighborhood $U(a, \delta)$.

In other words, $f \colon M \to \mathbb{R}$ is continuous at $a \in M$ iff

$$\forall\, \varepsilon\ \exists\, \delta \colon x \in M \wedge |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon\ .$$

Else we say that f is *discontinuous* at a. For example, $\operatorname{sgn}(x)$ is discontinuous at 0, but is continuous at every $x \neq 0$.

Proposition 5 (on continuity at a point) *Suppose that $b \in M \subset \mathbb{R}$, that b is a limit point of M and that a function $f\colon M \to \mathbb{R}$ is given. The following three claims are mutually equivalent.*

1. *The function f is continuous at the point b.*
2. $\lim_{x \to b} f(x) = f(b)$.
3. *For every sequence $(a_n) \subset M$ with $\lim a_n = b$ also $\lim f(a_n) = f(b)$.*

Proof. Implication $1 \Rightarrow 2$. We assume that f is continuous at b according to Definition 4 and that an ε is given. Thus there is a δ such that $f[U(b, \delta) \cap M] \subset U(f(b), \varepsilon)$. Thus also $f[P(b, \delta) \cap M] \subset U(f(b), \varepsilon)$ and, by the definition of limit of a function, $\lim_{x \to b} f(x) = f(b)$.

Implication $2 \Rightarrow 3$. We assume that $\lim_{x \to b} f(x) = f(b)$ and that a sequence $(a_n) \subset M$ with $\lim a_n = b$ is given, as well as an ε. By the definition of limit of a function there is a δ such that

$$f[P(b,\, \delta) \cap M] \subset U(f(b),\, \varepsilon)\;. \tag{*}$$

We take an n_0 such that $n \geq n_0 \Rightarrow a_n \in U(b, \delta)$. Hence $n \geq n_0 \Rightarrow f(a_n) \in U(f(b), \varepsilon)$: either $a_n \neq b$, and we can use inclusion (*), or $a_n = b$ but then $f(a_n) = f(b) \in U(f(b), \varepsilon)$. Thus $\lim f(a_n) = f(b)$.

Implication $3 \Rightarrow 1$, i.e., $\neg 1 \Rightarrow \neg 3$. We assume that f is not continuous at b according to Definition 4. Thus there is an ε such that for every δ there is an $a = a(\delta) \in U(b, \delta) \cap M$ with $f(a) \notin U(f(b), \varepsilon)$. We *choose* for every n some such $a_n :=$

$a(1/n)$ and get the sequence $(a_n) \subset M$ such that $\lim a_n = b$ but $f(a_n) \notin U(f(b), \varepsilon)$ for every n—$(f(a_n))$ does not have the limit $f(b)$. Therefore part 3 does not hold. $\square$

In the proof of the last implication we used again the so called *axiom of choice* of set theory.

We consider continuity of a function at a point that is not a limit point of the definition domain.

Definition 6 (isolated points) *A point $b \in M \subset \mathbb{R}$ is an isolated point of M if*

$$\exists\, \varepsilon:\ U(b,\, \varepsilon) \cap M = \{b\}\ .$$

For $b \in M \subset \mathbb{R}$ we see at once that

$$b \text{ is not a limit point of } M \iff b \text{ is an isolated point of } M\ .$$

Proposition 7 (continuity at an isolated point) *Let $b \in M \subset \mathbb{R}$, b be an isolated point of M and let*

$$f\colon M \to \mathbb{R}$$

be any function. Then f is continuous at b.

Proof. Let b, M and f be as stated. Then for some δ, $U(b, \delta) \cap M = \{b\}$. For this δ the inclusion

$$f[U(b,\, \delta) \cap M] = \{f(b)\} \subset U(f(b),\, \varepsilon)$$

holds for every ε. Hence f is continuous at b according to Definition 4. $\square$

So, for example, every sequence $(a_n) \subset \mathbb{R}$ when viewed as a function a from $\mathbb{N}$ to $\mathbb{R}$ is continuous at every point $n \in \mathbb{N} \subset \mathbb{R}$ of its definition domain $\mathbb{N}$.

- *One-sided continuity.* Let $a \in M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$. The function f is *left-continuous*, resp. *right-continuous*, at the point a if

$$\begin{aligned} \forall \varepsilon\, \exists \delta : \quad & f[U^-(a,\,\delta) \cap M] \subset U(f(a),\,\varepsilon)\ , \\ & \text{resp. } \ f[U^+(a,\,\delta) \cap M] \subset U(f(a),\,\varepsilon)\ . \end{aligned}$$

It is easy to see that

$$f \text{ is cont. at } a \iff f \text{ is left-cont. at } a \wedge f \text{ is right-cont. at } a\ .$$

- *The Riemann function.* This function

$$r\colon \mathbb{R} \to \{0\} \cup \{1/n \mid n \in \mathbb{N}\}$$

is defined by

$$r(x) = \begin{cases} 0 & \ldots \quad x \text{ is an irrational number and} \\ \frac{1}{n} & \ldots \quad x = \frac{m}{n} \in \mathbb{Q} \text{ and } \frac{m}{n} \text{ is in lowest terms}\ . \end{cases}$$

Proposition 8 (on $r(x)$) *The Riemann function is continuous at x if and only if x is irrational.*

Proof. Let $x = \frac{m}{n} \in \mathbb{Q}$, where $\frac{m}{n}$ is in lowest terms, and let $\varepsilon \le \frac{1}{n}$. For every δ there is an irrational number $\alpha \in U(x,\delta)$. But $r(\alpha) = 0 \notin U(r(x),\varepsilon) = U(\frac{1}{n},\varepsilon)$, and r is not continuous at the point x.

Let $x \in \mathbb{R}$ be irrational and let an $\varepsilon \in (0,1)$ be given. We define $\delta := \min(M)$ for the set

$$M := \{|x - \tfrac{m}{n}| \mid \tfrac{m}{n} \in \mathbb{Q},\, \tfrac{m}{n} \in U(x,\,1),\, 1/n \ge \varepsilon\}\ .$$

This $\delta > 0$ exists because $M \neq \emptyset$ and M is a finite set of positive numbers. (I explain this in detail orally.) Also $y \in U(x, \delta) \Rightarrow r(y) \in U(r(x), \varepsilon) = U(0, \varepsilon)$ because for every $y \in U(x, \delta)$ one has that $r(y) = 0$ or $r(y) = \frac{1}{n} < \varepsilon$. Therefore r is continuous at the point x. $\square$

• *Limits of monotonous functions.* Monotonicity of functions is similar to monotonicity of sequences.

Definition 9 (monotonous functions) *Let $M \subset \mathbb{R}$ and $f\colon M \to \mathbb{R}$. The function f*

1. *is non-decreasing (on M) if for every $x, y \in M$ one has that $x \leq y \Rightarrow f(x) \leq f(y)$, and*
2. *is non-increasing (on M) if for every $x, y \in M$ one has that $x \leq y \Rightarrow f(x) \geq f(y)$.*

The function f is monotonous (on M) if it is non-decreasing or non-increasing.

Recall when a set of real numbers is bounded from above (or from below) and when it is unbounded from above (or from below). We explain after the proof of the next theorem why it is stated only for one-sided limits and not for two-sided limits.

Theorem 10 (limits of monotonous functions) *Let $M \subset \mathbb{R}$, $a \in \mathbb{R}$ be a left limit point of M and let*

$$f\colon M \to \mathbb{R}$$

be a function that is non-decreasing on $P^-(a,\delta) \cap M$ for some δ. Then the left-sided limit of the function f at the point a exists. With $N := f[P^-(a,\delta) \cap M] \subset \mathbb{R}$ we have that

$$\lim_{x \to a^-} f(x) = \begin{cases} +\infty & \ldots \ N \text{ is unb. from above and} \\ \sup(N) \in \mathbb{R} & \ldots \ N \text{ is bounded from above}\,. \end{cases}$$

Proof. Suppose that N is unbounded from above and that an ε is given. Thus there is an $x \in P^-(a,\delta) \cap M$ such that $f(x) > 1/\varepsilon$. Since f is non-decreasing on $P^-(a,\delta) \cap M$, for $\theta := a - x$ it holds that $y \in P^-(a,\theta) \cap M \Rightarrow x < y < a \Rightarrow f(y) \geq f(x) > 1/\varepsilon$. Thus $f[P^-(a,\theta) \cap M] \subset U(+\infty,\varepsilon)$ and $\lim_{x \to a^-} f(x) = +\infty$.

Suppose that N is bounded from above, $s := \sup(N)$ and that an ε is given. By the definition of s there is an $x \in P^-(a,\delta) \cap M$ such that $s - \varepsilon < f(x) \leq s$. Since f is non-decreasing on $P^-(a,\delta) \cap M$, for $\theta := a - x$ it holds that $y \in P^-(a,\theta) \cap M \Rightarrow x < y < a \Rightarrow s - \varepsilon < f(x) \leq f(y) \leq s$. Hence $f[P^-(a,\theta) \cap M] \subset U(s,\varepsilon)$ and $\lim_{x \to a^-} f(x) = s$. $\square$

There are several other obvious variants of the theorem: for locally non-increasing functions and/or infinite limit points and/or right-sided limits. Existence of two-sided limits can be proven by monotonicity by reducing them to one-sided limits.

But monotonicity by itself does not guarantee existence of two-sided limits: consider the function $\mathrm{sgn}(x)\colon \mathbb{R} \to \{-1, 0, 1\}$ (recall that $\mathrm{sgn}(x) = -1$ for $x < 0$, $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(x) = 1$ for $x > 0$). It is monotonous (non-decreasing) on the whole $\mathbb{R}$, but

$$\lim_{x \to 0} \mathrm{sgn}(x)$$

does not exist, $\lim_{x \to 0^-} \mathrm{sgn}(x) = -1$ and $\lim_{x \to 0^+} \mathrm{sgn}(x) = 1$.

• *Arithmetic of limits of functions.* We state the next theorem for two-sided limits and prove it by means of Heine's definition of limits of functions. Fortunately now we need not estimate sums, products and ratios. Such estimates were dealt with in the proof of the theorem on arithmetic of limits of sequences.

Theorem 11 (arithmetic of limits of functions) *Let $M \subset \mathbb{R}$, $A, K, L \in \mathbb{R}^*$, A be a limit point of M and let the functions*

$$f,\, g\colon M \to \mathbb{R}$$

have limits $\lim_{x \to A} f(x) = K$ and $\lim_{x \to A} g(x) = L$. Then the following hold.

1. $\lim_{x \to A}(f(x) + g(x)) = K + L$ *whenever the right-hand side is defined.*
2. $\lim_{x \to A} f(x)g(x) = KL$ *whenever the right-hand side is defined.*
3. $\lim_{x \to A} f(x)/g(x) = K/L$ *whenever the right-hand side is defined. Here if $g(x) = 0$, $f(x)/g(x) := 0$.*

Proof. All proofs of 1–3 are similar and we therefore give in detail only the proof of 3. Let $(a_n) \subset M \setminus \{A\}$ be any sequence with

$\lim a_n = A$. By Heine's definition of limits of functions (implication $\Rightarrow$), $\lim f(a_n) = K$ a $\lim g(a_n) = L$. We assume that the right-hand side is defined (so that $L \neq 0$ and $g(a_n) \neq 0$ for every $n \geq n_0$). By the theorem on arithmetic of limits of sequences,

$$\lim \frac{f(a_n)}{g(a_n)} = \frac{\lim f(a_n)}{\lim g(a_n)} = \frac{K}{L} \ .$$

Since this holds for every sequence

$$(f(a_n)/g(a_n))$$

with (a_n) as above, by Heine's definition of limits of functions (implication $\Leftarrow$) also $\lim_{x \to A} f(x)/g(x) = K/L$. $\square$

There are obvious versions of the previous theorem for one-sided limits.

• *Limits of functions and order.* We give functional versions of the theorem on limits and order, and of the theorem on two cops. Recall that for $M, N \subset \mathbb{R}$ the comparison $M < N$ means that for every $a \in M$ and $b \in N$ one has that $a < b$.

Theorem 12 (limits of functions and order) *Let $A, K, L \in \mathbb{R}^*$, A be a limit point of $M \subset \mathbb{R}$ and let the functions*

$$f,\, g\colon M \to \mathbb{R}$$

have limits $\lim_{x\to A} f(x) = K$ and $\lim_{x\to A} g(x) = L$. The following hold.

1. *If $K < L$ then there is a δ such that $f[P(A,\delta)\cap M] < g[P(A,\delta)\cap M]$.*

2. *If for every δ there are $x, y \in P(A,\delta)\cap M$ with $f(x) \geq g(y)$, then $K \geq L$.*

Proof. 1. Since $K < L$, there is an ε such that $U(K,\varepsilon) < U(L,\varepsilon)$. Then by the assumption on limits of f and g there exists a δ such that $f[P(A,\delta)\cap M] \subset U(K,\varepsilon)$ and $g[P(A,\delta)\cap M] \subset U(L,\varepsilon)$. Hence

$$f[P(A,\delta)\cap M] < g[P(A,\delta)\cap M]\;.$$

2. We already know from the proof of this for sequences that part 2 is a reformulation of part 1. If part 1 is the implication $\varphi \Rightarrow \psi$, then part 2 is $\neg\psi \Rightarrow \neg\varphi$. $\square$

Recall that for $a, b \in \mathbb{R}$ we denote by $I(a,b)$ the closed real interval with endpoints a and b.

Theorem 13 (two functional cops) *Let* $A, L \in \mathbb{R}^*$*,* A *be a limit point of* $M \subset \mathbb{R}$ *and let functions*

$$f,\ g,\ h\colon M \to \mathbb{R}$$

be given such that $\lim_{x\to A} f(x) = \lim_{x\to A} h(x) = L$ *and that there is a* δ *such that for any* $x \in P(A,\delta) \cap M$*,* $g(x) \in I(f(x), h(x))$*. Then also*

$$\lim_{x\to A} g(x) = L\ .$$

Proof. Let A, L, M, f, g and h be as stated and let an ε be given. Thus there exists a δ such that the sets $f[P(A,\delta)\cap M]$ and $h[P(A,\delta)\cap M]$ are contained in $U(L,\varepsilon)$. Therefore and due to the convexity of the neighborhood $U(L,\varepsilon)$, for every $x \in P(A,\delta)\cap M$ one has that $I(f(x),h(x)) \subset U(L,\varepsilon)$. By the assumption one has that $g[P(A,\delta)\cap M] \subset U(L,\varepsilon)$, hence $\lim_{x\to A} g(x) = L$. $\square$

• *Limits of composite functions.* Composition of functions has no analogy for sequences. Therefore the next limit theorem on this operation is more interesting than the previous four theorems. After its proof we explain why our formulation is better than some other formulations.

Theorem 14 (limits of composite functions) *Let $A, K, L \in \mathbb{R}^*$, $M, N \subset \mathbb{R}$, A be a limit point of M and K a limit point of N, and let functions*

$$g\colon M \to N \ \text{ and } \ f\colon N \to \mathbb{R}$$

have limits $\lim_{x\to A} g(x) = K$ and $\lim_{x\to K} f(x) = L$. Then the composite function $f(g)\colon M \to \mathbb{R}$ has the limit

$$\lim_{x\to A} f(g)(x) = L$$

<u>if and only if</u> at least one the two conditions below holds.

1. *If $K \in N$ (so that $K \in \mathbb{R}$) then $f(K) = L$ (so that $L \in \mathbb{R}$).*
2. *There is a δ such that $K \notin g[P(A,\delta) \cap M]$.*

If neither 1 nor 2 holds then either $\lim_{x\to A} f(g)(x)$ does not exist or $\lim_{x\to A} f(g)(x) = f(K) \neq L$.

Proof. Let an ε be given. By the assumption on limits of f and g there is a δ such that (i) $f[P(K,\delta) \cap N] \subset U(L,\varepsilon)$, and a θ such that (ii) $g[P(A,\theta) \cap M] \subset U(K,\delta)$.

Condition 1 holds. Then inclusion (i) strengthens to $f[U(K,\delta) \cap N] \subset U(L,\varepsilon)$. Therefore in

$$f(g)[P(A,\,\theta)\cap M] = f[g[P(A,\,\theta)\cap M]] \subset f[U(K,\,\delta)\cap N] \subset U(L,\,\varepsilon)$$

the second inclusion holds and $\lim_{x\to A} f(g)(x) = L$.

Condition 2 holds. We take the θ smaller than the δ in Condition 2 and strengthen inclusion (ii) to $g[P(A,\theta) \cap M] \subset P(K,\delta)$.

Therefore in

$$f(g)[P(A,\,\theta)\cap M] = f[g[P(A,\,\theta)\cap M]] \subset f[P(K,\,\delta)\cap N] \subset U(L,\,\varepsilon)$$

the first inclusion holds and again $\lim_{x\to A} f(g(x)) = L$.

Neither condition 1 nor condition 2 holds. Then $K \in N$ but $f(K) \neq L$, and for every n there exists an $a_n \in P(A, 1/n) \cap M$ such that $g(a_n) = K$. Then the sequence $(a_n) \subset M \setminus \{A\}$, has the limit $\lim a_n = A$ and

$$\lim\ f(g)(a_n) = \lim\ f(g(a_n)) = \lim\ f(K) = f(K) \neq L\ .$$

By Heine's definition of limits of functions, either $\lim_{x\to A} f(g)(x)$ does not exist or $\lim_{x\to A} f(g)(x) = f(K) \neq L$. $\square$

If $K \notin N$, for example when $K = \pm\infty$, then Condition 1 always holds. Elsewhere Condition 1 is not formulated as an implication as here, but only as the requirement that $f(K) = L$. By our extension of Condition 1 here we have obtained the underlined equivalence. Another advantage of our formulation is that we say what happens if neither of the two conditions holds.

• *Asymptotic symbols O, o and $\sim$.* These are the most frequently used symbols denoting asymptotic relations between functions. One uses also symbols Θ, $\ll$, Ω and other.

Definition 15 (big O) *Let $M \subset \mathbb{R}$, $f, g\colon M \to \mathbb{R}$ and $N \subset M$. If*

$$\exists\, c > 0\ \forall\, x \in N\colon |f(x)| \le c \cdot |g(x)|\ ,$$

we write $f(x) = O(g(x))$ $(x \in N)$ and say that the function f is big O of the function g on the set N.

Examples.

1. Is $x^2 = O(x^3)$ $(x \in \mathbb{R})$? No, there is a problem at 0.
2. Is $x^3 = O(x^2)$ $(x \in \mathbb{R})$? No, there is a problem at infinities.
3. Is $x^3 = O(x^2)$ $(x \in (-20, 20))$? Yes.
4. Is $\log x = O(x^{1/3})$ $(x \in (0, +\infty))$? No, there is a problem at 0.
5. Is $\log x = O(x^{1/3})$ $(x \in (1, +\infty))$? Yes.

The remaining two asymptotic symbols are defined by means of limits.

Definition 16 (little o and $\sim$) *Let $A \in \mathbb{R}^*$ be a limit point of $M \subset \mathbb{R}$, let $f, g\colon M \to \mathbb{R}$ and let $g \neq 0$ on $P(A, \delta) \cap M$ for some δ.*

1. *If $\lim_{x \to A} f(x)/g(x) = 0$, we write $f(x) = o(g(x))$ $(x \to A)$ and say that the function f is little o of g when x goes to A.*
2. *If $\lim_{x \to A} f(x)/g(x) = 1$, we write $f(x) \sim g(x)$ $(x \to A)$ and say that the function f is asymptotically equal to g when x goes to A.*

Examples.

1. Is $x^2 = o(x^3)$ $(x \to +\infty)$? Yes.
2. Is $x^3 = o(x^2)$ $(x \to 0)$? Yes.
3. Is $x^2 = o(x^3)$ $(x \to 0)$? No.

4. Is $(x+1)^3 \sim x^3$ $(x \to 1)$? No, the ratio goes to 2.
5. Is $(x+1)^3 \sim x^3$ $(x \to +\infty)$? Yes.
6. Is $e^{-1/x^2} = o(x^{20})$ $(x \to 0)$? No, e^{-1/x^2} goes to 0 faster than any x^n.

THANK YOU FOR YOUR ATTENTION!