LECTURE 4, 3/9/2022

MORE ON SERIES. LIMITS OF FUNCTIONS. ELEMENTARY FUNCTIONS

• *Infinite series.* Recall that the symbols

$$\sum a_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots$$

of a *series* denote the sequence $(a_n) \subset \mathbb{R}$, whose terms a_n are now called *summands*, and also the limit

$$\lim\, s_n = \lim_{n\to\infty} (a_1 + a_2 + \dots + a_n) \in \mathbb{R}^*$$

of the sequence

$$(s_n) = (a_1 + a_2 + \dots + a_n)$$

of *partial sums* s_n, which is called the *sum* (of the series). If the sum is finite we say that the series *converges*, else it *diverges*. Convergence and divergence of any series do not depend on any change of only finitely many summands but, in contrast with limits of sequences, the sum may change after the change of a single summand.

We keep the indices in sequences (a_n) to be $n \in \mathbb{N}$, so that $(a_n) = (a_1, a_2, \dots)$, but for series the summation index n often runs through sets different from $\mathbb{N}$ and often we use for it other letters. So one can encounter series like

$$\sum_{m=0}^{\infty} a_m, \;\; \sum_{j=6}^{100} b_j, \;\; \sum_{n=-\infty}^{+\infty} a_n z^n, \;\; \sum_{\substack{n\in A\\ n\neq x}} u_n, \;\; \sum_{k\geq 0} c_k\ ,$$

not speaking of double and multiple series. The next result follows at once from the theorem on monotone sequences.

Proposition 1 (nonnegative summands) *Every series $\sum a_n$ whose summands $a_n \geq 0$ for every $n \geq n_0$, has a sum that differs from $-\infty$.*

A similar proposition holds for series with almost all summands non-positive.

Proposition 2 (necessary condition of convergence) *If the series $\sum a_n$ converges then* $\lim a_n = 0$.

Proof. If $\sum a_n$ converges then $\lim s_n =: S \in \mathbb{R}$ (here $s_n = \sum_{j=1}^{n} a_j$). By limits of subsequences and by the arithmetic of limits,

$$\lim a_n = \lim (s_n - s_{n-1}) = \lim s_n - \lim s_{n-1} = S - S = 0 \; .$$

□

By this proposition both series

$$\textstyle\sum_{n=1}^{\infty} 1 = 1 + 1 + \dots \quad \text{and} \quad \sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$$

diverge. The former has the sum $+\infty$ (see Proposition 1) and the latter (mentioned at the end of the last lecture) does not have a sum.

• *Harmonic series.* In the previous proposition the opposite implication does not hold. We consider the series with the summands

$$a_1 = \frac{1}{2},\ a_2 = a_3 = \frac{1}{4},\ a_4 = a_5 = a_6 = a_7 = \frac{1}{8},\ \dots$$
$$\dots,\ a_{2^k} = a_{2^k+1} = \dots = a_{2^{k+1}-1} = \frac{1}{2^{k+1}},\ \dots \; .$$

Clearly, $\lim a_n = 0$, but $s_1 < s_2 < \dots$ and

$$s_{2^{k+1}-1} = \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + \dots + 2^k \cdot \frac{1}{2^{k+1}} = \frac{k+1}{2} \; ,$$

so that $\sum a_n = \lim s_n = +\infty$ (why?) and the series diverges. Thus we have the following result.

Proposition 3 (harmonic series) *So called* harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

diverges and has the sum $+\infty$.

Proof. Let (h_n) be the partial sums of the harmonic series and (s_n) be the partial sums of the previous series $\sum a_n$. Then $1/n > a_n$ for every n, therefore also $h_n > s_n$ for every n. Since $\lim s_n = +\infty$, the one cop theorem gives that $\lim h_n = +\infty$ and the harmonic series has the sum $+\infty$. $\square$

Partial sums of the harmonic series are called *harmonic numbers*. We mention without proof two interesting results on them.

Theorem 4 (on harmonic numbers) *We consider the harmonic numbers* $h_n = \sum_{j=1}^{n} 1/j$, $n \in \mathbb{N}$.

1. *For every* $n \in \mathbb{N}$,

$$h_n = \log n + \gamma + \Delta_n \ ,$$

   where $\gamma = 0.57721\dots$ *is so called Euler's constant and the numbers* $\Delta_n \in \mathbb{R}$ *satisfy that* $|\Delta_n| < c/n$ *for a constant* c *and every* n.

2. $h_n \in \mathbb{N} \iff n = 1$.

The conjecture that $\gamma \notin \mathbb{Q}$ is still unproven.

• *The Riemann theorem.* At the beginning of the 1st lecture we met in the paradox of infinite sums the series

$$1 - 1 + \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{3} + \cdots + \tfrac{1}{n} - \tfrac{1}{n} + \ldots$$

that has an "obvious" sum 0. By changing the order of summands we changed this sum to a positive one. The original sum 0 is correct, though, because the series has partial sums $1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots$ going in limit to 0.

Theorem 5 (Riemann's) *Let $\sum_{n=1}^{\infty} a_n$ be a series of the same type, i.e.,*

1. $\lim a_n = 0$*,*
2. $\sum a_{k_n} = +\infty$*, where a_{k_n} are positive summands of the series, and*
3. $\sum a_{z_n} = -\infty$*, where a_{z_n} are negative summands of the series.*

Then $\forall\, S \in \mathbb{R}^*$ there is a bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = S$$

— by changing the order of summands we can get any sum. There is of course also a bijection π such that the series $\sum_{n=1}^{\infty} a_{\pi(n)}$ does not have a sum.

The theorem is named after the German mathematician *Bernhard Riemann (1826–1866)*. He also invented an integral of real functions which we will study in this course later.

• *Absolutely convergent series.* We introduce a class of series whose sums do not change under reordering of summands.

Definition 6 (AC series) *A series $\sum a_n$ is absolutely convergent, abbreviated* AC, *if the series $\sum |a_n|$ converges.*

The class of AC series is the correct generalization of finite sums to infinitely many summands.

Proposition 7 (on AC series) *Every* AC *series converges.*

Proof. Let $\sum a_n$ be an AC series and (s_n) be its partial sums. We show that (s_n) is a Cauchy sequence. This suffices because by the theorem on Cauchy condition then (s_n) converges. For every two indices $m \le n$ we have that

$$\begin{aligned} |s_n - s_m| &= |a_{m+1} + a_{m+2} + \cdots + a_n| \\ &\overset{\Delta\text{-ineq.}}{\le} |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| = t_n - t_m = |t_n - t_m| \;, \end{aligned}$$

where (t_n) are partial sums of the series $\sum |a_n|$. But the sequence (t_n) is Cauchy (by the mentioned theorem) and therefore also (s_n) is Cauchy. $\square$

Theorem 8 (commutativity of AC series) *If $\sum a_n$ is an* AC *series, then for every bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ the series $\sum a_{\pi(n)}$ is* AC. *The sums of the original and reordered series are equal,*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\pi(n)} \;.$$

- *Geometric series.* These are the series

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + \cdots + q^n + \ldots$$

with the parameter $q \in \mathbb{R}$ called the *quotient.*

Theorem 9 (on geometric series) *For $q \leq -1$ the geometric series does not have a sum. For $-1 < q < 1$ the geometric series converges and has the sum*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \ .$$

For $q \geq 1$ the geometric series has the sum $+\infty$.

Proof. For every $q \in \mathbb{R} \setminus \{1\}$ and every $n \in \mathbb{N}$,

$$s_n := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q} = \frac{1}{1-q} + \frac{q^n}{q-1} \ .$$

So for $q < -1$ we have by the arithmetic of limits that $\lim\ s_{2n-1} = +\infty$, $\lim\ s_{2n} = -\infty$ and therefore $\lim\ s_n$ does not exist — the geometric series does not have a sum. For $q = -1$ we have similarly that $s_{2n-1} = 1$, $s_{2n} = 0$ and the geometric series again does not have a sum. For $-1 < q < 1$ one has that $\lim\ q^n = 0$ and by the arithmetic of limit the geometric series has the sum $\lim\ s_n = \frac{1}{1-q}$. For $q = 1$ one has that $s_n = n$ and the geometric series has the sum $\lim\ s_n = +\infty$. For $q > 1$, $\lim\ q^n = +\infty$ and by the arithmetic of limits the geometric series has the sum $\lim\ s_n = +\infty$. □

A quick application of the formula for the sum of geometric series:

$$27.272727\cdots = 27(1 + 10^{-2} + 10^{-4} + \ldots) = 27 \cdot \frac{1}{1 - 10^{-2}}$$
$$= \frac{27 \cdot 100}{99} = \frac{300}{11}\,.$$

It is easy to see that for $q \in (-1, 1)$ and $m \in \mathbb{Z}$ one has the more general formula

$$q^m + q^{m+1} + q^{m+2} + \cdots = \frac{q^m}{1 - q}\,.$$

It is also clear that every convergent geometric series is absolutely convergent.

• *Zeta function* $\zeta(s)$. This is a function $\zeta(s)\colon \mathbb{C}\backslash\{1\} \to \mathbb{C}$ defined by a series. Here we define it only for real $s > 1$. We use real powers a^b for $a > 0$ which will be defined in the second half of this lecture. So for $s \in \mathbb{R}$ we take the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\,.$$

Theorem 10 (on zeta function) *For $s \le 1$ the series $\zeta(s)$ has the sum $+\infty$. For $s > 1$ it (absolutely) converges.*

The first claim follows from the divergence of harmonic series. L. Euler derived formulas for all values $\zeta(2n)$ for every n, for example $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$. No formula is known for $\zeta(2n-1)$ for any $n \ge 2$. It is known that $\zeta(3) \notin \mathbb{Q}$.

• *Limits of functions.* For any $A \in \mathbb{R}^*$ and any $\varepsilon > 0$, recall the ε-neighborhood $U(A, \varepsilon)$ of A and the deleted ε-neighborhood $P(A, \varepsilon) = U(A, \varepsilon) \setminus \{A\}$ of A.

Definition 11 (limit points) *We say that $L \in \mathbb{R}^*$ is a limit point of a set $M \subset \mathbb{R}$ if $\forall\, \varepsilon \colon P(L, \varepsilon) \cap M \neq \emptyset$.*

In other words, $L \in \mathbb{R}^*$ is a limit point of a set $M \subset \mathbb{R}$ if and only if there is a sequence $(a_n) \subset M \setminus \{L\}$ with $\lim a_n = L$. Now we generalize the notion of limit from sequences to functions. Recall that for $f \colon A \to B$ and $C \subset A$, $f[C] = \{f(x) \mid x \in C\} \subset B$.

Definition 12 (limits of functions) *Let $A, L \in \mathbb{R}^*$, $M \subset \mathbb{R}$, A be a limit point of M and let $f \colon M \to \mathbb{R}$ be a function. If*

$$\forall\, \varepsilon\; \exists\, \delta \colon f[P(A,\, \delta) \cap M] \subset U(L,\, \varepsilon)\ ,$$

we write $\lim_{x \to A} f(x) = L$ and say that the function f has at A the limit L.

The limit does not depend on the value $f(A)$ and f need not, and for $A = \pm\infty$ even cannot, be defined at A. For a sequence $(a_n) \subset \mathbb{R}$,

$$\lim a_n = \lim_{x \to +\infty} a(x)\ ,$$

where on the right-hand side we understand the sequence as a function $a \colon \mathbb{N} \to \mathbb{R}$. When A is not a limit point of M then for some δ one has that $M \cap P(A, \delta) = \emptyset$. Then

$$\emptyset = f[P(A,\, \delta) \cap M] \subset U(L,\, \varepsilon)$$

pro *every* $L \in \mathbb{R}^*$ *and every* ε, which is not good.

Proposition 13 (uniqueness of limits) *Limits of functions are unique: if $M \subset \mathbb{R}$, $f\colon M \to \mathbb{R}$, $K, L, L' \in \mathbb{R}^*$ and K is a limit point of the set M, then*

$$\lim_{x\to K} f(x) = L \wedge \lim_{x\to K} f(x) = L' \Rightarrow L = L' \ .$$

Proof. We prove it directly, like for limits of sequences. For every ε there is a δ such that the nonempty set $f[P(K,\delta) \cap M]$ is contained in both neighborhoods $U(L,\varepsilon)$ and $U(L',\varepsilon)$. In particular, $\forall\, \varepsilon\colon U(L,\varepsilon) \cap U(L',\varepsilon) \neq \emptyset$. Thus (by the main property of neighborhoods mentioned earlier) $L = L'$. $\square$

The next theorem shows how to reduce limits of functions to limits of sequences.

Theorem 14 (Heine's definition) *Let $M \subset \mathbb{R}$, K, L be in $\mathbb{R}^*$, K be a limit point of the set M and let $f\colon M \to \mathbb{R}$. Then*

$$\lim_{x\to K} f(x) = L \iff$$
$$\iff \forall\, (a_n) \subset M \setminus \{K\}:\ \lim a_n = K \Rightarrow \lim f(a_n) = L \ .$$

Thus L is the limit of the function f at K iff for every sequence (a_n) in M that has the limit K but never equals K, the values $(f(a_n))$ have the limit L.

Proof. Implication $\Rightarrow$. We assume that $\lim_{x\to K} f(x) = L$, that $(a_n) \subset M \setminus \{K\}$ has the limit K and that an ε is given. Then there exists a δ such that for every $x \in M \cap P(K,\delta)$ one has that $f(x) \in U(L,\varepsilon)$. For this δ there is an n_0 such that $n \geq n_0 \Rightarrow a_n \in$

$P(K,\delta)\cap M$. Hence $n\ge n_0\Rightarrow f(a_n)\in U(L,\varepsilon)$ and $f(a_n)\to L$.

Implication $\neg\Rightarrow\neg$. We assume that $\lim_{x\to K}f(x)=L$ does not hold and deduce from this that the right-hand side of the equivalence does not hold. So there is an $\varepsilon>0$ such that for every $\delta>0$ there is a point $b=b(\delta)\in M\cap P(K,\delta)$ such that $f(b)\not\in U(L,\varepsilon)$. We set $\delta=\frac{1}{n}$ for $n\in\mathbb{N}$ and for every n *chose* a point $b_n:=b(1/n)\in M\cap P(K,1/n)$ such that $f(b_n)\not\in U(L,\varepsilon)$. The sequence (b_n) lies in $M\setminus\{K\}$ and has the limit K, but the sequence of values $(f(b_n))$ does not have the limit L. The right-hand side of the equivalence therefore does not hold. $\square$

In the proof of the implication $\Leftarrow$ we used the so called *axiom of choice* from the set theory.

One example on the limit of a function: due to the identity $a^2-b^2=(a-b)(a+b)$,

$$\begin{aligned}\lim_{x\to+\infty}\left(\sqrt{x+\sqrt{x}}-\sqrt{x}\right) &\overset{\frac{(\sqrt{\dots}-\sqrt{x})\cdot(\sqrt{\dots}+\sqrt{x})}{\sqrt{\dots}+\sqrt{x}}}{=} \lim_{x\to+\infty}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x}}+\sqrt{x}}\\ &\overset{\frac{\dots/\sqrt{x}}{\dots/\sqrt{x}}}{=} \lim_{x\to+\infty}\frac{1}{\sqrt{1+1/\sqrt{x}}+1}\\ &= \frac{1}{1+1}=\frac{1}{2}\,.\end{aligned}$$

• *The exponential function.* This is the most important elementary function.

Definition 15 (the exponential) *For any $x\in\mathbb{R}$ we set*

$$\mathrm{e}^x=\exp(x):=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2}+\frac{x^3}{6}+\dots:\mathbb{R}\to\mathbb{R}\,.$$

This series is AC for every real (even complex) x, due to an estimate by geometric series: $|x/n| < 1$ whenever $n > |x|$.

Proposition 16 (the exponential identity) *For every* $x, y \in \mathbb{R}$*,*

$$\exp(x + y) = \exp(x) \cdot \exp(y)\ .$$

Proposition 17 (on the exponential function) *It holds that*

1. $\exp(0) = 1$*,*
2. $\forall\, x \in \mathbb{R} : \exp(x) > 0 \wedge \exp(-x) = 1/\exp(x)$*,*
3. $\exp$ *increases,* $x < y \Rightarrow \exp(x) < \exp(y)$*,*
4. $\lim_{x\to-\infty} \exp(x) = 0$*,*
5. $\lim_{x\to+\infty} \exp(x) = +\infty$ *and*
6. $\exp$ *is a bijection from* $\mathbb{R}$ *to* $(0, +\infty)$*.*

Definition 18 (the number e**)** *We define* $\mathrm{e} := \exp(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = 2.71828\ldots$*, it is called the Euler number.*

It is not very hard to show that e is irrational, $\mathrm{e} \notin \mathbb{Q}$.

The *logarithm* $\log x$ is the inverse to the exponential function,

$$\log := \exp^{-1} \colon (0, +\infty) \to \mathbb{R}\ .$$

Its basic properties derive from those of the exponential function.

Proposition 19 (on logarithm) *It holds that*

1. $\log(1) = 0$,
2. $\forall\, x, y \in (0, +\infty) : \log(xy) = \log x + \log y$,
3. $\log$ *increases*, $x < y \Rightarrow \log(x) < \log(y)$,
4. $\lim_{x\to 0} \log(x) = -\infty$,
5. $\lim_{x\to +\infty} \log(x) = +\infty$ *a*
6. $\log$ *is a bijection from* $(0, +\infty)$ *to* $\mathbb{R}$.

• *The real power* a^b. Here we introduce only the simplified version with nonnegative a. But everybody knows that, for example, $(-2)^3 = (-2) \cdot (-2) \cdot (-2) = -8$.

Definition 20 (real power) *For* $a, b \in \mathbb{R}$ *with* $a > 0$ *we set*

$$a^b := \exp(b \log a)\ .$$

For every $b > 0$ *we set* $0^b := 0$.

For the number $\mathrm{e} = \exp(1)$ and every $x \in \mathbb{R}$ then indeed $\mathrm{e}^x = \exp(x \log(\exp(1))) = \exp(x \cdot 1) = \exp(x)$.

Proposition 21 (3 power identities) *For any numbers* $a, b, x, y \in \mathbb{R}$ *with* $a, b > 0$,

$$(a \cdot b)^x = a^x \cdot b^x, \quad a^x \cdot a^y = a^{x+y} \quad \& \quad (a^x)^y = a^{x\cdot y}\ .$$

Proof. 1. $(ab)^x = \exp(x \log(ab)) = \exp(x \log a + x \log b) = \exp(x \log a) \exp(x \log b) = a^x b^x$.

2. $a^x a^y = \exp(x\log a)\exp(y\log a) = \exp(x\log a + y\log a) = \exp((x+y)\log a) = a^{x+y}$.

3. $(a^x)^y = \exp(y\log(\exp(x\log a))) = \exp(yx\log a) = a^{xy}$. $\square$

But note that

$$\left((-1)^2\right)^{1/2} = 1^{1/2} = 1 \neq -1 = (-1)^1 = (-1)^{2\cdot 1/2} .$$

The power 0^0 is problematic because of the following reason.

Proposition 22 (0^0 is indeterminate) *For every number $c \in [0,1]$ there exist sequences (a_n), $(b_n) \subset (0,+\infty)$ such that*

$$\lim a_n = \lim b_n = 0 \quad \textit{and} \quad \lim\,(a_n)^{b_n} = c\ .$$

Both sequences can be also selected so that $\lim\,(a_n)^{b_n}$ does not exist.

• *Cosine and sine.* These functions can be defined by infinite series too but their origin lies in geometry.

Definition 23 (cosine and sine) *For every $t \in \mathbb{R}$ we define the functions*

$$\cos t := \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \quad \textit{and} \quad \sin t := \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}\ ,$$

so that $\cos t = 1 - \frac{t^2}{2} + \frac{t^4}{24} - \dots$ and $\sin t = t - \frac{t^3}{6} + \frac{t^5}{120} - \dots$, going from $\mathbb{R}$ do $\mathbb{R}$.

Again by geometric series estimates we see that both series are AC for every $t \in \mathbb{R}$. We frame the basic property of cosine and sine in the track and field terminology.

Theorem 24 (on a runner) *Let $t \in \mathbb{R}$ and*

$$S := \{(x,\, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

be the plane unit circle (i.e., with radius 1) with center in the origin. The runner that runs on the track S with unit speed, starts at the point $(1, 0) \in S$ and runs counter-clockwisely for $t > 0$ and clockwisely for $t \leq 0$, is in the time $|t|$ located in the point

$$(\cos t,\, \sin t) \in S\ .$$

Thus cosine and sine coincide with the geometricly defined functions bearing the same names.

Definition 25 (the number π) *We can informally define $\pi = 3.14159\ldots$ so that the circumference of S, i.e., the time when the runner again runs through the start, equals 2π. The formal definition is that the smallest positive zero of the function $\cos t$ is $\pi/2$.*

The definition by the circumference of S is informal because (in these slides) we do not posses any precise definition of the length of a circular arc. Here are the basic properties of cosine and sine.

Proposition 26 (on sine and cosine) *It holds that*

1. *cosine and sine are* 2π*-periodic functions,* $\cos(t+2\pi) = \cos t$ *and* $\sin(t+2\pi) = \sin t$ *for every* $t \in \mathbb{R}$*,*
2. *sine increases on* $[0, \pi/2]$ *from* 0 *to* 1*,*
3. $\forall\, t \in [0, \pi]\colon\ \sin(t) = \sin(\pi - t)$ *and* $\forall\, t \in [0, 2\pi]\colon\ \sin(t) = -\sin(2\pi - t)$*,*
4. $\forall\, t \in [0, 2\pi]\colon\ \cos t = \sin(t + \pi/2)$*,*
5. $\forall\, t \in \mathbb{R}\colon\ \cos^2 t + \sin^2 = 1$ *and*
6. $\forall\, s, t \in \mathbb{R}\colon\ \sin(s \pm t) = \sin s \cdot \cos t \pm \cos s \cdot \sin t$ *and* $\cos(s \pm t) = \cos s \cdot \cos t \mp \sin s \cdot \sin t$*.*

Parts 2–4 imply that $\cos, \sin\colon \mathbb{R} \to [-1, 1]$. Part 4 says that the graph of cosine is just the shifted graph of sine.

Further trigonometric functions are the *tangent* $\tan t = \frac{\sin t}{\cos t}$ and the *cotangent* $\cot t = \frac{\cos t}{\sin t}$. The *arcsine* (*inverse sine*) and the *arccosine* (*inverse cosine*) is the inverse of the restriction of sine and cosine to the interval $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. They are the bijections

$$\arcsin\colon [-1,\, 1] \to [-\pi/2,\, \pi/2] \quad \text{and} \quad \arccos\colon [-1,\, 1] \to [0,\, \pi]\ .$$

Similarly, the *arctangent* and the *arccotangent* is the inverse of the restriction of tangent and cotangent to the interval $(-\pi/2, \pi/2)$ and $(0, \pi)$, respectively. They are the bijections

$$\arctan\colon \mathbb{R} \to (-\pi/2,\, \pi/2) \quad \text{and} \quad \operatorname{arccot}\colon \mathbb{R} \to (0,\, \pi)\ .$$

THANK YOU FOR YOUR ATTENTION