LECTURE 3, 3/2/2022
ARITHMETIC OF LIMITS. LIMITS AND ORDER. INFINITE SERIES

• *Arithmetic of limits.* Last time we considered existence of limits of real sequences. Now we look at relations between limits and arithmetical operations, and between limits and ordering. Recall that (a_n), (b_n) and (c_n) denote real sequences and that $\mathbb{R}^*$ is the extended real line. Recall how to compute with infinities. The *variant form* of the Δ-inequality $|a+b| \le |a| + |b|$ is that

$$|a-b| \ge |a| - |b|\ .$$

The next theorem is useful for finding limits. In its proof we use a reformulation of existence of finite limits: if $(a_n) \subset \mathbb{R}$ and $a \in \mathbb{R}$ then

$$\lim a_n = a \iff a_n =: a + \underbrace{e_n}_{\text{error term}} \ \text{ where } e_n \to 0$$

(so $e_n = a_n - a$).

Theorem 1 (arithmetic of limits). *Let* $\lim a_n = K \in \mathbb{R}^*$ *and* $\lim b_n = L \in \mathbb{R}^*$. *Then*

1. $\lim(a_n + b_n) = K + L$ *whenever the right-hand side is defined,*

2. $\lim a_n b_n = KL$ *whenever the right-hand side is defined and*

3. $\lim a_n/b_n = K/L$ *whenever the right-hand side is defined. For* $b_n = 0$ *we set* $a_n/b_n := 0$.

RS in 1 is not defined $\iff$ $K = -L = \pm\infty$. *RS in 2 is not defined* $\iff$ $K = 0$ *and* $L = \pm\infty$ *or* $K = \pm\infty$ *and* $L = 0$. *RS in 3 is not defined* $\iff$ $L = 0$ *or* $K = \pm\infty$ *and* $L = \pm\infty$.

Proof. 1. Let $K, L \in \mathbb{R}$ and an ε be given. There is an n_0 such that $n \geq n_0 \Rightarrow a_n =: K + c_n$ and $b_n =: L + d_n$ with $|c_n|, |d_n| < \frac{\varepsilon}{2}$.

Thus $n \geq n_0 \Rightarrow a_n + b_n = K + L + \overbrace{c_n + d_n}^{e_n}$ with $|e_n| \leq |c_n| + |d_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $a_n + b_n \to K + L$.

Let $K \neq -\infty$, $L = +\infty$ and a c be given. Then $a_n > d$ for every n and some d, and $b_n > -d + c$ for every $n \geq n_0$. Thus $n \geq n_0 \Rightarrow a_n + b_n > d + (-d + c) = c$ and $a_n + b_n \to +\infty$. The case that $K = -\infty$ and $L \neq +\infty$ is similar.

2. Let $K, L \in \mathbb{R}$ and an $\varepsilon \in (0, 1)$ be given. There is an n_0 such that $n \geq n_0 \Rightarrow a_n =: K + c_n$ and $b_n =: L + d_n$ with $|c_n|, |d_n| < \varepsilon$.

Thus $n \geq n_0 \Rightarrow a_n b_n = KL + \overbrace{c_n L + d_n K + c_n d_n}^{e_n}$ and

$$|e_n| \overset{\Delta\text{-ineq.}}{\leq} \varepsilon(|K| + |L| + 1) \to 0 \;\; \text{pro} \;\; \varepsilon \to 0 \,.$$

So $a_n b_n \to KL$.

Let $K > 0$, $L = -\infty$ and a $c < 0$ be given. Then $a_n > d > 0$ for every $n \geq n_0$ and some $d > 0$, and $b_n < c/d$ for every $n \geq n_0$. Thus $n \geq n_0 \Rightarrow a_n b_n < d(c/d) = c$ and $a_n b_n \to -\infty$. The other cases with $K = \pm\infty$ or $L = \pm\infty$ are similar.

3. Let $K, L \in \mathbb{R}$ with $L \neq 0$ and an ε be given. There is an n_0 such that $n \geq n_0 \Rightarrow a_n =: K + c_n$ and $b_n =: L + d_n$ with $|c_n| < \varepsilon$ and $|d_n| < \min(\varepsilon, |L|/2)$. For every $n \geq n_0$ we then have that

$$\frac{a_n}{b_n} = \frac{K + c_n}{L + d_n} = \frac{K/L + c_n/L}{1 + d_n/L} = \frac{K}{L} \underbrace{- \frac{K d_n / L^2}{1 + d_n/L} + \frac{c_n/L}{1 + d_n/L}}_{e_n}$$

and, due to $|1 + d_n/L| \geq 1 - |d_n|/|L| \geq 1 - 1/2 = 1/2$,

$$|e_n| \overset{\Delta\text{-ineq. and its variant}}{\leq} \frac{|K|\varepsilon/L^2}{1/2} + \frac{\varepsilon/|L|}{1/2} = \varepsilon \cdot \left(\frac{2|K|}{L^2} + \frac{2}{|L|} \right) \to 0$$

for $\varepsilon \to 0$. Thus $a_n/b_n \to K/L$.

Let $K \in \mathbb{R}$, $L = -\infty$ and an ε be given. Hence (a_n) is bounded, $|a_n| < c$ for every n and some $c > 0$, and there is an n_0 such that $n \geq n_0 \Rightarrow b_n < -c/\varepsilon$. Hence $n \geq n_0 \Rightarrow |a_n/b_n| < c/|b_n| < c/(c/\varepsilon) = \varepsilon$ and $a_n/b_n \to 0$. The other cases when $L \neq 0$ and either $K = \pm\infty$ or $L = \pm\infty$ are similar. $\square$

The theorem of course does not give complete characterization of arithmetic of limits. Even when its assumptions are not met, i.e.,

K or L does not exist or the right-hand side is not defined, the (unique) limit on the left-hand side may still exist. Below we list several such cases without proof.

Proposition 2 (supplement 1) *Even when $K = \lim a_n$ does not exist, the following hold.*

1. *(a_n) bounded and $L = \lim b_n = \pm\infty \Rightarrow \lim(a_n + b_n) = L$.*
2. *(a_n) bounded and $L = \lim b_n = 0 \Rightarrow \lim a_n b_n = 0$.*
3. *(a_n) satisfies $a_n > c > 0$ for $n \geq n_0$ and $L = \lim b_n = \pm\infty \Rightarrow \lim a_n b_n = L$.*
4. *(a_n) bounded and $L = \lim b_n = \pm\infty \Rightarrow \lim a_n/b_n = 0$.*
5. *(a_n) satisfies $a_n > c > 0$ for $n \geq n_0$, $b_n > 0$ for $n \geq n_0$ and $L = \lim b_n = 0 \Rightarrow \lim a_n/b_n = +\infty$.*

But often it indeed happens that when the assumptions of the theorem are not satisfied, the limit on the left-hand side is not uniquely determined or does not exist.

Proposition 3 (supplement 2) *For every $A \in \mathbb{R}^*$ there exist sequences (a_n), (b_n) such that*

1. $\lim a_n = +\infty$, $\lim b_n = -\infty$ *and* $\lim (a_n + b_n) = A$,
2. $\lim a_n = 0$, $\lim b_n = \pm\infty$ *and* $\lim a_n b_n = A$ *and*
3. $\lim a_n = \lim b_n = 0$ *or* $\lim a_n = \pm\infty$, $\lim b_n = \pm\infty$ *and* $\lim a_n/b_n = A$.

The limits $\lim (a_n + b_n)$, $\lim a_n b_n$ *and* $\lim a_n/b_n$ *in 1–3 also need not exist.*

• *Sequences given by recurrences.* We meet the first real limits of sequences, $\lim (n^{1/3} - n^{1/2})$, $\lim \frac{2n-3}{5n+4}$ etc. we saw earlier are in reality problems on limits of functions. We explain how to compute limits of recurrent sequences in the next proof. We use in it so called *AG inequality* (the inequality between arithmetic and geometric mean): for every two real numbers $a, b \geq 0$,

$$\frac{a+b}{2} \geq \sqrt{ab}\ .$$

Proposition 4 (recurrent limit). *Let (a_n) be given by $a_1 = 1$ and, for $n \geq 2$,*

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}\ .$$

Then $\lim a_n = \sqrt{2}$.

Proof. Suppose that $L := \lim a_n \in \mathbb{R}$ exists and is finite. Since limits are preserved by subsequences, $\lim a_{n-1} = L$. By parts 3, 2 and 1 of the previous theorem we have that $\lim \frac{1}{a_{n-1}} = \frac{1}{L}$ for $L \neq 0$, always $\lim \frac{a_{n-1}}{2} = \frac{L}{2}$ and $\lim \left(\frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}\right) = \frac{L}{2} + \frac{1}{L}$ for $L \neq 0$. Thus

$$L = \frac{L}{2} + \frac{1}{L} \rightsquigarrow L^2 - L^2/2 = 1 \rightsquigarrow L^2 = 2$$

and we have two solutions $L = \sqrt{2}$ and $L = -\sqrt{2}$. If we prove that (a_n) converges, we get that $\lim a_n = \sqrt{2}$ because $a_n > 0$ for every n and therefore $L \geq 0$ (as we see in the next part of the lecture).

However, to exclude that $L = 0$ we need an inequality stronger than $L \geq 0$. But next we show that $a_n \geq \sqrt{2}$ for every $n \geq 2$. Thus $L \geq \sqrt{2} > 0$, if L exists, and certainly $L \neq 0$.

In order that we can use the theorem on monotone sequences from the last lecture, we show that (a_n) is non-increasing from $n_0 = 2$. So we need that for every $n \geq 2$,

$$a_n \geq a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \iff \frac{a_n^2}{2} \geq 1 \iff a_n \geq \sqrt{2}\ .$$

But for $n \geq 2$ the AG inequality indeed shows that

$$a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}} = \frac{a_{n-1} + 2a_{n-1}^{-1}}{2} \geq \sqrt{a_{n-1} \cdot 2a_{n-1}^{-1}} = \sqrt{2}\ .$$

Hence (a_n) is non-increasing from $n_0 = 2$ and non-negative, so bounded from below. By the theorem on monotone sequences, (a_n) has a non-negative finite limit. Thus $\lim a_n = \sqrt{2}$. $\square$

The initial computation, i.e., solving the equation obtained by replacing all $a_n, a_{n-1}, \ldots$ in the recurrence with the putative limit L, is of any value only if we show that (a_n) converges. For instance, the recurrence sequence (a_n) defined by $a_1 = 1$ and $a_n = -a_{n-1}$ does *not* have the limit $\lim a_n = 0$ although the equation $L = -L$

has a unique solution $L = 0$, because $(a_n) = (1, -1, 1, -1, \dots)$ does not have a limit (as we noted earlier).

In the proof of the next proposition we use the simple observation that

$$\lim a_n = 0 \iff \lim |a_n| = 0 \, .$$

Indeed, $a_n \to 0 \iff \forall \varepsilon \, \exists n_0 \colon \; n \geq n_0 \Rightarrow |a_n| < \varepsilon \iff \forall \varepsilon \, \exists n_0 \colon \; n \geq n_0 \Rightarrow |\,|a_n|\,| < \varepsilon \iff |a_n| \to 0$.

Proposition 5 (geometric sequences) *For $q \in \mathbb{R}$ the limit*

$$\lim_{n\to\infty} q^n \begin{cases} = 0 & \dots \;\; |q| < 1, \; \textit{i.e.}, \; -1 < q < 1, \\ = 1 & \dots \;\; q = 1, \\ = +\infty & \dots \;\; q > 1 \; \textit{and} \\ \textit{does not exist} & \dots \;\; q \leq -1 \, . \end{cases}$$

Proof. 1. Let $|q| < 1$. By the observation we may assume that $q \geq 0$. Then (q^n) is non-increasing, bounded from below (since $q^n \geq 0$) and by the theorem on monotone sequences it has a non-negative finite limit L. From $q^n = q \cdot q^{n-1}$ we get the equation $L = q \cdot L \leadsto L = 0/(1-q) = 0$.

2. For $q = 1$ we have the constant sequence $(1, 1, \dots)$ that has the limit 1.

3. Let $q > 1$. By part 1 of this proposition and by part 5 of proposition 2,

$$\lim_{n\to\infty} q^n = \lim_{n\to\infty} \frac{1}{(1/q)^n} = \frac{1}{0^+} = +\infty \, .$$

4. Let $q \leq -1$. For $q = -1$, $(q^n) = (-1, 1, -1, 1, \dots)$ does not have a limit because it has a subsequence with limit 1, and

a subsequence with limit -1. For $q < -1$, (q^n) does not have a limit because by part 3 of this proposition and by arithmetic of limits it has a subsequence with limit $+\infty$, and a subsequence with limit $-\infty$. $\square$

• *Limits and* $(\mathbb{R}^*, <)$*.* Relations between limits of real sequences and the linear order $(\mathbb{R}^*, <)$ are described in the next two theorems.

Theorem 6 (lim and order). *Suppose that $K, L \in \mathbb{R}^*$ and that $(a_n), (b_n)$ are two real sequences with* $\lim a_n = K$ *and* $\lim b_n = L$*. The following hold.*

1. *If $K < L$ then there is an n_0 such that for every two (possibly distinct!) indices $m, n \geq n_0$ one has that $a_m < b_n$.*
2. *If for every n_0 there are indices m and n such that $m, n \geq n_0$ and $a_m \geq b_n$, then $K \geq L$.*

Proof. 1. Let $K < L$. As we know from the last lecture, there is an ε such that $U(K, \varepsilon) < U(L, \varepsilon)$. By the definition of limit there is an n_0 such that $m, n \geq n_0 \Rightarrow a_m \in U(K, \varepsilon)$ and $b_n \in U(L, \varepsilon)$. So $m, n \geq n_0 \Rightarrow a_m < b_n$.

2. We get the proof of this for free by elementary logic: the implication $\varphi \Rightarrow \psi$ is equivalent with the variant $\neg\psi \Rightarrow \neg\varphi$. But the variant of the implication in part 1 is exactly part 2. $\square$

Strict inequality between terms of two sequences may turn in limit in equality of their limits: for $(a_n) := (1/n)$ and $(b_n) := (0, 0, \dots)$ we have that $a_m > b_n$ for every m and n, but

$$\lim a_n = \lim b_n = 0 \ .$$

The previous theorem is often (in fact, almost always) presented in the weaker form that if $K < L$ then there is an n_0 such that $n \geq n_0 \Rightarrow a_n < b_n$. Similarly for part 2.

For $a, b \in \mathbb{R}$ we denote by $I(a, b)$ the interval with endpoints a and b:

$$I(a, b) = [a, b] \text{ for } a \leq b \text{ and } I(a, b) = [b, a] \text{ for } a \geq b \ .$$

A set $M \subset \mathbb{R}$ is *convex* if $\forall\, a, b \in M\colon I(a, b) \subset M$.

Proposition 7 (on intervals) *Convex sets of real numbers are exactly: $\emptyset$, the singletons $\{a\}$ for $a \in \mathbb{R}$, the whole $\mathbb{R}$ and the intervals (a, b), $(-\infty, a)$,*

$$(a,\, +\infty),\ (a,\, b],\ [a,\, b),\ [a,\, b],\ (-\infty,\, a] \ \textit{ and } \ [a,\, +\infty)$$

for real numbers $a < b$.

Every neighborhood $U(A, \varepsilon)$ is convex. No deleted neighborhood $P(a, \varepsilon)$ is convex.

The next theorem is popular because of its name.

Theorem 8 (two cops theorem). *Let $a \in \mathbb{R}$ and (a_n), (b_n) and (c_n) be three real sequences such that*

$$\lim\, a_n = \lim\, c_n = a \wedge \forall\, n \geq n_0\colon b_n \in I(a_n,\, c_n) \ .$$

Then $\lim\, b_n = a$ *too.*

Proof. Let a, (a_n), (b_n) and (c_n) be as stated and an ε be given. By the definition of limit there is an n_0 such that $n \geq n_0 \Rightarrow a_n, c_n \in U(a, \varepsilon)$. Since $U(a, \varepsilon)$ is convex, $n \geq n_0 \Rightarrow I(a_n, c_n) \subset U(a, \varepsilon)$.

Due to the assumption we have that $n \geq n_0 \Rightarrow b_n \in U(a, \varepsilon)$ and $b_n \to a$. $\square$

Two cops, the sequences (a_n) and (c_n), lead a suspect, the sequence (b_n), to the common limit a. For infinite limit, one cop suffices: if $\lim a_n = -\infty$ and $b_n \leq a_n$ for every $n \geq n_0$, then also $\lim b_n = -\infty$. Similarly for the limit $+\infty$. The two cops theorem is often presented in a weaker form, with inequalities $a_n \leq b_n \leq c_n$ in place of the membership $b_n \in I(a_n, c_n)$. Then the cops are firmly positioned to the left and right sides of the suspect, whereas in our version of the theorem they are allowed to exchange their places.

• *Limes inferior and limes superior of a sequence.* These are residues of Latin mathematical terminology which mean "the least limit" and "the largest limit", respectively.

Definition 9 (limit point) *Let $A \in \mathbb{R}^*$ and $(a_n) \subset \mathbb{R}$. We say that A is a limit point of the sequence (a_n) if $\lim a_{m_n} = A$ for a subsequence (a_{m_n}) of (a_n). We set*

$$H(a_n) := \{A \in \mathbb{R}^* \mid A \text{ is a limit point of } (a_n)\} \subset \mathbb{R}^* \,.$$

Limes inferior of a sequence (a_n), denoted $\liminf a_n$, is defined as $\min(H(a_n))$ in the linear order $(\mathbb{R}^*, <)$. *Limes superior* of the sequence, denoted $\limsup a_n$, is the element $\max(H(a_n))$. In the next theorem we show that these elements exist.

Theorem 10 (liminf and limsup exist) *For every real sequence (a_n), the set $H(a_n)$ is nonempty and it possesses in the linear order $(\mathbb{R}^*, <)$ both minimum and maximum element.*

Proof. Let (a_n) be a real sequence. Last time we proved that (a_n) has a subsequence with a limit, so that $H(a_n) \neq \emptyset$. We prove the existence of $\max(H(a_n))$, for the minimum element one proceeds similarly.

In the following four cases, which cover all possibilities, we define an element $A \in \mathbb{R}^*$. (i) If $H(a_n) = \{-\infty\}$ then $A := -\infty$. (ii) If $+\infty \in H(a_n)$ then $A := +\infty$. (iii) If $H(a_n) \cap \mathbb{R} \neq \emptyset$ and this set is unbounded from above then $A := +\infty$. (iv) Finally, if $+\infty \notin H(a_n)$ and the set $H(a_n) \cap \mathbb{R}$ is nonempty and bounded from above, then

$$A := \sup(H(a_n) \cap \mathbb{R}) \in \mathbb{R} \ .$$

We show that always $A = \max(H(a_n))$. In the cases (i) and (ii) it clearly holds. In the cases (iii) a (iv) it is clear that $A \geq h$ for every $h \in H(a_n)$ and it suffices to show that $A \in H(a_n)$. In the cases (iii) and (iv) it is also clear that there is a sequence

$$(b_n) \subset H(a_n) \cap \mathbb{R} \text{ such that } \lim b_n = A \ .$$

Since every number b_n is the limit of a subsequence of (a_n), we easily find a subsequence (a_{m_n}) such that

$$\forall\, n \colon a_{m_n} \in U(b_n,\, 1/n) \ .$$

But then $\lim a_{m_n} = \lim b_n = A$ and $A \in H(a_n)$. $\square$

Theorem 11 (properties of liminf and limsup). *For any real sequence (a_n) the following hold.*

1. *If* $\lim\, a_n$ *exists then* $H(a_n) = \{\lim\, a_n\}$.

2. *Three exclusive cases occur and cover all possibilities: (i)* (a_n) *is unbounded from above and* $\limsup\, a_n = +\infty$, *(ii)* $\lim\, a_n = -\infty$ *and* $\limsup\, a_n = -\infty$, *(iii)* $\limsup\, a_n$ *is finite and*
$$\limsup\, a_n = \lim_{n\to\infty} \big(\sup(\{a_m \mid m \geq n\})\big) \in \mathbb{R}\ .$$

3. *Three exclusive cases occur and cover all possibilities: (i)* (a_n) *is unbounded from below and* $\liminf\, a_n = -\infty$, *(ii)* $\lim\, a_n = +\infty$ *and* $\liminf\, a_n = +\infty$, *(iii)* $\liminf\, a_n$ *is finite and*
$$\liminf\, a_n = \lim_{n\to\infty} \big(\inf(\{a_m \mid m \geq n\})\big) \in \mathbb{R}\ .$$

4. *Always* $\liminf\, a_n \leq \limsup\, a_n$ *and equality holds if and only if* $\lim\, a_n$ *exists and then*
$$\liminf\, a_n = \limsup\, a_n = \lim\, a_n\ .$$

Proof. 1. This is obvious, any subsequence of a sequence with a limit has the same limit.

2. The first two cases are more or less clear. Suppose neither of them occurs. For every n we set $A_n := \{a_m \mid m \geq n\}$ and $b_n := \sup(A_n)$. Every set A_n is bounded from above and nonempty, so that (b_n) is a well defined real sequence that is obviously non-increasing. By the theorem on monotone sequences it has a limit

$L := \lim b_n \in \mathbb{R} \cup \{-\infty\}$. Clearly, $L \neq -\infty$ for else we would have that $\lim a_n = -\infty$. Hence $L \in \mathbb{R}$. By the definition of supremum,

$$\forall\, n\, \exists\, m\ (\geq n) :\ b_n - 1/n < a_m \leq b_n\ .$$

It follows from this that $\lim b_n = L \in H(a_n)$. Suppose that L is not the maximum of $H(a_n)$. Then there is a $\delta > 0$ such that for infinitely many m one has that $a_m > L + \delta$. Then we can take an n such that $b_n < L + \delta$. But then there would be an $m \geq n$ such that $a_m > L + \delta > b_n$, in contradiction with the definition of b_n. Thus $L = \max(H(a_n)) = \limsup a_n$.

3. Proof of this is very similar to the previous proof.

4. The first claim is clear. To prove the second one it suffices to prove that if $\liminf a_n = \limsup a_n =: L$ then $\lim a_n = L$. When $L = \pm\infty$, $\lim a_n = L$ by case (ii) in part 2 or part 3. Let $L \in \mathbb{R}$ and an ε be given. By case (iii) in parts 2 and 3 we take an n such that

$$L - \varepsilon < \inf(\{a_m \mid m \geq n\}) \leq \sup(\{a_m \mid m \geq n\}) < L + \varepsilon\ .$$

Then $m \geq n \Rightarrow L - \varepsilon < a_m < L + \varepsilon$ so that $a_n \to L$. $\square$

• *Infinite series.* We introduce basic notions of the theory of (infinite) series. You will hear more about series next time.

Definition 12 (infinite series) *An (infinite) series is again a sequence $(a_n) \subset \mathbb{R}$. Its sum is the limit*

$$\sum a_n = \sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots := \lim\,(a_1 + a_2 + \cdots + a_n)$$

if it exists. The terms in the sequence $(a_1 + a_2 + \cdots + a_n)$ are so called partial sums (of the series).

The symbols $\sum a_n$, $\sum_{n=1}^{\infty} a_n$ and $a_1 + a_2 + \ldots$ are, however, often used to denote also the sequence (a_n) itself. We met infinite series in the first lecture in the first paradox. Is it true that

$$\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots = 0 + 0 + 0 + \cdots = 0\ ?$$

No, this is not true. The first equality holds, it is an equality between two sequences. The third equality holds as well, it says that the sum of all zeros series is zero. But the second equality does not hold: as an equality of two sequences it does not hold and neither it holds as an equality of sums of two series, because the series $1 - 1 + 1 - 1 + \ldots$ does not have any sum, the sequence of partial sums $(1, 0, 1, 0, \ldots)$ does not have a limit.

THANK YOU FOR YOUR ATTENTION