LECTURE 2, 2/23/2022
EXISTENCE THEOREMS FOR LIMITS OF SEQUENCES

• *Review.* Recall the real numbers $\mathbb{R}$ and recall the natural numbers $\mathbb{N} = \{1, 2, \dots\}$. We denote the latter by the letters i, j, k, l, m, m_0, m_1, $\dots$, n, n_0, n_1, $\dots$. The letters a, b, c, d, e, δ, ε and θ, possibly with indices, denote real numbers. Always $\delta, \varepsilon, \theta > 0$ and we think of them as close to 0. Recall that $(a_n) \subset \mathbb{R}$ is a real sequence.

• *Computing with infinities.* For the general notion of a limit we add to $\mathbb{R}$ the *infinities* $+\infty$ and $-\infty$. We get the *extended real axis*

$$\mathbb{R}^* := \mathbb{R} \cup \{+\infty, -\infty\} \ .$$

We compute with infinities according to the following rules.

We always take only all upper or all lower signs:

$$\begin{aligned} A \in \mathbb{R} \cup \{\pm\infty\} \ &\Rightarrow \ A + (\pm\infty) = \pm\infty + A := \pm\infty \ , \\ A \in (0,\, +\infty) \cup \{+\infty\} \ &\Rightarrow \ A \cdot (\pm\infty) = (\pm\infty) \cdot A := \pm\infty \ , \\ A \in (-\infty,\, 0) \cup \{-\infty\} \ &\Rightarrow \ A \cdot (\pm\infty) = (\pm\infty) \cdot A := \mp\infty \ , \\ a \in \mathbb{R} \ &\Rightarrow \ \frac{a}{\pm\infty} := 0 \ , \end{aligned}$$

$-(\pm\infty) := \mp\infty$, $-\infty < a < +\infty$ and $-\infty < +\infty$.

Subtraction of an element $A \in \mathbb{R}^*$ reduces to adding $-A$ and division by $a \neq 0$ means multiplication by $1/a$. All remaining values of the operations, that is ($A \in \mathbb{R}^*$)

$$\frac{A}{0},\ (\pm\infty) + (\mp\infty),\ 0 \cdot (\pm\infty),\ (\pm\infty) \cdot 0,\ \frac{\pm\infty}{\pm\infty} \ \text{ and } \ \frac{\pm\infty}{\mp\infty} \ ,$$

are undefined, these are so called *indeterminate expressions*. Elements of $\mathbb{R}^*$ are usually denoted by A, B, K and L.

• *Neighborhoods of points and infinities.* We remind the notation for real intervals:

$$(a,\, b] = \{x \in \mathbb{R} \mid a < x \le b\},\ (-\infty,\, a) = \{x \in \mathbb{R} \mid x < a\}$$

etc.

Definition 1 (neighborhoods) *For any $\varepsilon > 0$, the ε-neighborhood of a point b and the deleted ε-neighborhood of b is defined, respectively, as*

$$U(b,\, \varepsilon) := (b-\varepsilon,\, b+\varepsilon) \ \text{ and } \ P(b,\, \varepsilon) := (b-\varepsilon,\, b) \cup (b,\, b+\varepsilon)\ ,$$

so that $P(b, \varepsilon) = U(b, \varepsilon) \setminus \{b\}$. An ε-neighborhood of infinity is

$$U(-\infty,\, \varepsilon) := (-\infty,\, -1/\varepsilon) \ \text{ and } \ U(+\infty,\, \varepsilon) := (1/\varepsilon,\, +\infty)\ .$$

We set $P(\pm\infty, \varepsilon) := U(\pm\infty, \varepsilon)$.

The *main property* of neighborhoods is that if $V, V' \in \{U, P\}$ then

$$A,\, B \in \mathbb{R}^*,\, A < B \Rightarrow \exists\, \varepsilon :\ V(A,\, \varepsilon) < V'(B,\, \varepsilon)\ ,$$

i.e., $a < b$ for every $a \in V(A, \varepsilon)$ and every $b \in V'(B, \varepsilon)$. In particular, $A \neq B \Rightarrow \exists\, \varepsilon :\ V(A, \varepsilon) \cap V'(B, \varepsilon) = \emptyset$.

• *Limits of sequences.* By $(a_n), (b_n), (c_n) \subset \mathbb{R}$ we denote real sequences. The next definition belongs to fundamental ones in analysis (and in mathematics).

Definition 2 (limit of a sequence) *Let* (a_n) *be a real sequence and* $L \in \mathbb{R}^*$. *If*

$$\forall \varepsilon \, \exists n_0 : \; n \geq n_0 \Rightarrow a_n \in U(L, \varepsilon) \, ,$$

we write that $\lim a_n = L$ *and say that the sequence* (a_n) *has the limit* L.

For $L \in \mathbb{R}$ we speak of a *finite* limit, and for $L = \pm\infty$ of an *infinite* limit. Sequences with finite limits *converge*, else they *diverge*. If $\lim a_n = a \in \mathbb{R}$ then for every real (and arbitrarily small) $\varepsilon > 0$ there is an index $n_0 \in \mathbb{N}$ such that for every index $n \in \mathbb{N}$ at least n_0 the distance between a_n and a is smaller than ε:

$$|a_n - a| < \varepsilon \, .$$

If $\lim a_n = -\infty$ then for every (negative) $c \in \mathbb{R}$ there is an index n_0 such that for every index n at least n_0,

$$a_n < c \, .$$

Similarly, with the inequality reversed, for the limit $+\infty$. We will use also the notation $\lim_{n\to\infty} a_n = L$ and $a_n \to L$. The simplest convergent sequence is the *eventually constant* sequence (a_n) with $a_n = a$ for every $n \geq n_0$, then of course $\lim a_n = a$. The popular image of a limit that "a sequence gets closer and closer to the limit but never reaches it (possibly only in infinity)", is a poetic one but is incorrect.

Proposition 3 (uniqueness of lim**)** *Limits are unique,* $\lim a_n = K$ *and* $\lim a_n = L \Rightarrow K = L$.

Proof. Let $\lim a_n = K$, $\lim a_n = L$ and let an ε be given. By Definition 2 there is an n_0 such that $n \geq n_0 \Rightarrow a_n \in U(K, \varepsilon)$ and $a_n \in U(L, \varepsilon)$. Thus $\forall\, \varepsilon :\ U(K, \varepsilon) \cap U(L, \varepsilon) \neq \emptyset$. By the main property of neighborhoods mentioned above, $K = L$. $\square$

• *Two limits.* We show that $\lim \frac{1}{n} = 0$. It is clear because for every ε and every $n \geq n_0 := 1 + \lceil 1/\varepsilon \rceil$,

$$0 < \frac{1}{n} \leq \frac{1}{\underbrace{1 + \lceil 1/\varepsilon \rceil}_{> 1/\varepsilon}} < \frac{1}{1/\varepsilon} = \varepsilon \rightsquigarrow 1/n \in U(0,\, \varepsilon)\ .$$

Here $\lceil a \rceil \in \mathbb{Z}$ denotes the *upper integral part* of the number a, the least $v \in \mathbb{Z}$ such that $v \geq a$. Similarly, the *lower integral part* $\lfloor a \rfloor$ of the number a is the largest $v \in \mathbb{Z}$ such that $v \leq a$. Our second example is that

$$\sqrt[3]{n} - \sqrt{n} \to -\infty\ .$$

Indeed, for any given $c < 0$ and every $n \geq n_0 > \max(4c^2, 2^6)$,

$$\overbrace{\sqrt[3]{n} - \sqrt{n}}^{\text{non-trivial}} = \overbrace{n^{1/2} \cdot \underbrace{(n^{-1/6} - 1)}_{n > 2^6 \Rightarrow \cdots < -1/2}}^{\text{trivial}} < \underbrace{-n^{1/2}}_{\cdots < -2|c|} /2 < -2|c|/2 = c\ .$$

It is not necessary to find an optimum n_0 in terms of ε or c. This is easy to do only in the simplest cases like $\lim \frac{1}{n}$, and else it may be complicated. It fully suffices to have some value n_0 such that for every $n \geq n_0$ the inequality (i.e., the membership) in the definition of limit holds. But to achieve it one still needs some skill in manipulating inequalities and estimates.

• *Subsequences of sequences.*

Definition 4 (subsequence) *A sequence* (b_n) *is a subsequence of a sequence* (a_n) *if there is a sequence (of natural numbers)* $m_1 < m_2 < \ldots$ *such that for every* n*,*

$$b_n = a_{m_n} \; .$$

We will use the notation that $(b_n) \preceq (a_n)$*.*

It is clear that the relation $\preceq$ on the set of sequences is reflexive and transitive. It is easy to find sequences (a_n) and (b_n) such that $(a_n) \preceq (b_n)$ and $(b_n) \preceq (a_n)$ but $(a_n) \neq (b_n)$.

Proposition 5 ($\preceq$ preserves limits) *Let* $(b_n) \preceq (a_n)$ *and let* $\lim a_n = L \in \mathbb{R}^*$*. Then also* $\lim b_n = L$*.*

Proof. It follows at once from Definitions 2 and 4 because the sequence (m_n) in the last definition has the property that $m_n \geq n$ for every n. □

The following useful proposition holds. Later we prove part 1 of it.

Proposition 6 (on subsequences) *Let (a_n) be a real sequence and let $A \in \mathbb{R}^*$. The following hold.*

1. *There is a sequence (b_n) such that $(b_n) \preceq (a_n)$ and (b_n) has a limit.*
2. *The sequence (a_n) does not have a limit $\iff$ (a_n) has two subsequences with different limits.*
3. *It is not true that $\lim a_n = A \iff$ there is a sequence (b_n) such that $(b_n) \preceq (a_n)$ and (b_n) has a limit different from A.*

Therefore we can always refute that a sequence has a limit by exhibiting two subsequences of it that have different limits. For example,

$$(a_n) := ((-1)^n) = (-1,\ 1,\ -1,\ 1,\ -1,\ \dots)$$

does not have a limit because $(1, 1, \dots) \preceq (a_n)$ and $(-1, -1, \dots) \preceq (a_n)$.

• *The limit of the n-th root of n.* One should be able to recognize when the computation of the given limit is "trivial" and when it is "non-trivial". The former is the case when in the expression whose limit one computes no two growths fight each other, else the latter case occurs. For instance, to compute the limits $\lim\,(2^n + 3^n)$ and $\lim \frac{4}{5n-3}$ is trivial, but to compute the limits $\lim\,(2^n - 3^n)$ and $\lim \frac{4n+7}{5n-3}$ is non-trivial. Often we compute a non-trivial limit by transforming the expression algebraically in a trivial form, like in the above example with $\sqrt[3]{n} - \sqrt{n}$. The next limit of $n^{1/n}$ is non-trivial because $n \to +\infty$ but $1/n \to 0$ and $(+\infty)^0$ is another indeterminate expression. We will see that the exponent prevails and $n^{1/n} \to 1$.

Proposition 7 **($n^{1/n} \to 1$)** *It holds that*

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1\ .$$

Proof. Always $n^{1/n} \ge 1$. If $n^{1/n} \not\to 1$, there would be a number $c > 0$ and a sequence $2 \le n_1 < n_2 < \dots$ such that for every i one has that $n_i^{1/n_i} > 1 + c$. By the Binomial Theorem we would have for every i that

$$\begin{aligned} n_i \ &> \ (1+c)^{n_i} = \textstyle\sum_{j=0}^{n_i} \binom{n_i}{j} c^j = 1 + \binom{n_i}{1} c + \binom{n_i}{2} c^2 + \dots + \binom{n_i}{n_i} c^{n_i} \\ &\ge \ \tfrac{n_i(n_i-1)}{2} \cdot c^2 \end{aligned}$$

and so, for every i,

$$n_i > \frac{n_i(n_i-1)}{2} \cdot c^2 \rightsquigarrow 1 + \frac{2}{c^2} > n_i\ .$$

This is a contradiction, the sequence $n_1 < n_2 < \dots$ cannot be upper-bounded. □

• *When a sequence has a limit.* We present four theorems (9, 10, 13 and 15) in this spirit, the second one will not be included in the exam. It is clear that existence of the limit of a sequence and its value are not influenced by changing only finitely many terms in the sequence. Thus properties ensuring existence of limits should be also *robust* in this sense, they should be independent of changes of finitely many terms in the sequence. For instance boundedness of sequences, which we define later, is a robust property. The following theorem on monotone sequences is often stated only for sequences

(a_n) monotone for every n, which is not a robust property. In the mentioned four theorems we employ robust properties.

- *Monotone (or monotonous) sequences.*

Definition 8 (monotonicity) *A sequence* (a_n) *is*

- *non-decreasing if* $a_n \leq a_{n+1}$ *for every* n*,*
- *non-decreasing from* n_0 *if* $a_n \leq a_{n+1}$ *for every* $n \geq n_0$*,*
- *non-increasing if* $a_n \geq a_{n+1}$ *for every* n*,*
- *non-increasing from* n_0 *if* $a_n \geq a_{n+1}$ *for every* $n \geq n_0$*,*
- *monotonous if it is non-decreasing or non-increasing,*
- *monotonous from* n_0 *if it is non-decreasing from* n_0 *or non-increasing from* n_0*.*

The inequalities $a_n < a_{n+1}$*, respectively* $a_n > a_{n+1}$*, yield a (strictly) increasing, respectively a (strictly) decreasing, sequence.*

A sequence (a_n) is *bounded from above* (BFA) if $\exists\, c\; \forall\, n:\; a_n < c$, else (a_n) is *unbounded from above* (UFA). Taking the reverse inequality we get *boundedness*, resp. *unboundedness*, of (a_n) *from below* (BFB and UFB). The sequence is *bounded*, if it is bounded both from above and from below. Each of these five properties of sequences is robust.

Theorem 9 (on monotone sequences) *Any real sequence* (a_n) *that is monotone from* n_0 *has a limit. If* (a_n) *is non-decreasing from* n_0 *then*

$$\lim_{n\to\infty} a_n = \begin{cases} \sup(\{a_n \mid n \geq n_0\}) & \ldots \ (a_n) \textit{ is } \text{BFA} \textit{ and} \\ +\infty & \ldots \ (a_n) \textit{ is } \text{UFA}. \end{cases}$$

If (a_n) *is non-increasing from* n_0 *then*

$$\lim_{n\to\infty} a_n = \begin{cases} \inf(\{a_n \mid n \geq n_0\}) & \ldots \ (a_n) \textit{ is } \text{BFB} \textit{ and} \\ -\infty & \ldots \ (a_n) \textit{ is } \text{UFB}. \end{cases}$$

Proof. We consider only the first case of a sequence that is non-decreasing from n_0, the other case is similar. If (a_n) is unbounded from above then for any given c there exists an m such that $a_m > \max(c, a_1, a_2, \ldots, a_{n_0})$. Thus $a_m > c$ and $m > n_0$. Therefore for every $n \geq m$,

$$a_n \geq a_{n-1} \geq \cdots \geq a_m > c \rightsquigarrow a_n > c$$

and $a_n \to +\infty$.

For (a_n) bounded from above we set $s := \sup(\{a_n \mid n \geq n_0\})$. Suppose that an $\varepsilon > 0$ is given. By the definition of supremum there exists an $m \geq n_0$ such that $s - \varepsilon < a_m \leq s$. Thus for every $n \geq m$,

$$s - \varepsilon < a_m \leq \cdots \leq a_{n-1} \leq a_n \leq s \rightsquigarrow s - \varepsilon < a_n \leq s$$

and $a_n \to s$. □

• *Quasi-monotonous sequences (not included in the exam).* We say that a sequence (a_n) is *quasi-monotone from* n_0 if

$$n \geq n_0 \Rightarrow \text{every set } \{m \mid a_m < a_n\} \text{ is finite}$$

or

$$n \geq n_0 \Rightarrow \text{every set } \{m \mid a_m > a_n\} \text{ is finite .}$$

Clearly, any sequence monotonous from an n_0 is quasi-monotonous from the same n_0. It is not hard to devise a sequence that is not monotonous from n_0 for any n_0, but is quasi-monotonous from some n_0.

In the next theorem we use the quantities lim sup and lim inf of a sequence. They are always defined, may attain values $\pm\infty$ and will be introduced in the next lecture.

Theorem 10 (on quasi-mon. sequences) *Every sequence $(a_n) \subset \mathbb{R}$ that is quasi-monotonous from n_0 has a limit. If (a_n) satisfies the 1st, resp. the 2nd, condition in the definition, then*

$$\lim\, a_n = \limsup a_n \in \mathbb{R}^*, \ \textit{resp.}\ \lim\, a_n = \liminf a_n \in \mathbb{R}^* \ .$$

Proof. We consider only the case that (a_n) satisfies the 1st condition for some n_0, the other case is similar. We suppose that (a_n) is unbounded from above and that a c is given. Hence there is an $m \geq n_0$ such that $a_m > c$. By the 1st condition there exist a k such that $a_n \geq a_m > c$ for every $n \geq k$. Thus $a_n \to +\infty = \limsup\, a_n$. Suppose that (a_n) is bounded from above, that $s := \limsup\, a_n \in \mathbb{R}$ and that an ε is given. By the definition of $\limsup\, a_n$, in

$$s - \varepsilon < a_m < s + \varepsilon$$

the first inequality holds for infinitely many m and the second one for almost all m. By the 1st condition there exists a k such that $s - \varepsilon < a_n < s + \varepsilon$ holds for every $n \geq k$. Thus $a_n \to s$. $\square$

Quasi-monotonous sequences, in which $n_0 = 1$, were introduced by the English mathematician *Godfrey H. Hardy (1877–1947)*.

• *The Bolzano–Weierstrass theorem.* For its proof we need the next result that is of independent interest.

Proposition 11 (existence of mon. subsequences) *Any sequence of real numbers has a monotonous subsequence.*

Proof. For a given (a_n) we consider the set

$$M := \{n \mid \forall\, m :\ n \le m \Rightarrow a_n \ge a_m\}\ .$$

If it is infinite, $M = \{m_1 < m_2 < \ldots\}$, we have the non-increasing subsequence (a_{m_n}). If M is finite, we take a number $m_1 > \max(M)$. Then certainly $m_1 \notin M$ and there is a number $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. As $m_2 \notin M$, there is an $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. And so on, we get a non-decreasing, even strictly increasing, subsequence (a_{m_n}). □

The theorem on monotone sequences and the previous proposition have the following two immediate corollaries. The first one is part 1 of Proposition 6.

Corollary 12 (subsequence with a limit) *Any real sequence has a subsequence that has a limit.*

Theorem 13 (Bolzano–Weierstrass) *Any bounded sequence of real numbers has a convergent subsequence.*

Proof. Let (a_n) be a bounded sequence and $(b_n) \preceq (a_n)$ be its monotonous subsequence guaranteed by the previous proposition. It is clear that (b_n) is bounded and by Theorem 9 it has a finite limit. $\square$

Karl Weierstrass (1815–1897) was a German mathematician, he was the "father of the modern mathematical analysis". The priest, philosopher and mathematician *Bernard Bolzano (1781–1848)* had Italian, German and Czech roots. In Prague there is a street named after him (near Hlavní nádraží), in the Celetná street a plaque commemorates him and his grave is in Olšanské hřbitovy (cemetery).

- *The Cauchy condition.*

Definition 14 (Cauchy sequences) *A sequence $(a_n) \subset \mathbb{R}$ is Cauchy if*

$$\forall\,\varepsilon\;\exists\,n_0:\;\; m,\,n \ge n_0 \Rightarrow |a_m - a_n| < \varepsilon\;,$$

i.e., $a_m \in U(a_n, \varepsilon)$.

The property that a sequence of real numbers is Cauchy is a robust one. It is clear that every Cauchy sequence is bounded.

Theorem 15 (Cauchy condition) *A sequence $(a_n) \subset \mathbb{R}$ converges if and only if (a_n) is Cauchy.*

Proof. The implication $\Rightarrow$. Let $\lim a_n = a$ and let an ε be given. Then there is an n_0 such that $n \ge n_0 \Rightarrow |a_n - a| < \varepsilon/2$. Thus

$$m,\,n \ge n_0 \Rightarrow |a_m - a_n| \le |a_m - a| + |a - a_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and (a_n) is a Cauchy sequence. (We have used that $a_m - a_n = (a_m - a) + (a - a_n)$ and that the triangle inequality $|c+d| \leq |c| + |d|$ holds.)

The implication $\Leftarrow$. Let (a_n) be a Cauchy sequence. We know that (a_n) is bounded, and therefore by the Bolzano–Weierstrass theorem it has a convergent subsequence (a_{m_n}) with a limit a. For a given ε we have an n_0 such that $n \geq n_0 \Rightarrow |a_{m_n} - a| < \varepsilon/2$ and that $m, n \geq n_0 \Rightarrow |a_m - a_n| < \varepsilon/2$. Always $m_n \geq n$ and therefore

$$n \geq n_0 \Rightarrow |a_n - a| \leq |a_n - a_{m_n}| + |a_{m_n} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon \; .$$

Thus $a_n \to a$. $\square$

Also the French mathematician *Augustin-Louis Cauchy (1789–1857)* lived in Prague, in political exile in 1833–1838.

THANK YOU FOR YOUR ATTENTION