LAST LECTURE 13, 5/18/2022

THE RIEMANN INTEGRAL AND ITS UPGRADE THE HENSTOCK–KURZWEIL INTEGRAL. USE OF INTEGRALS

• *The Riemann integral after J.-G. Darboux.* We give another equivalent definition of the Riemann integral. For real numbers $a < b$ and for a partition $P = (a_0, a_1, \dots, a_k)$ of the interval $[a, b]$ we denote $I_i := [a_{i-1}, a_i]$ and $|I_i| := a_i - a_{i-1}$. For a function $f\colon [a, b] \to \mathbb{R}$, the sums

$$s(P,\ f) := \sum_{i=1}^{k} |I_i| \cdot \inf(f[I_i]) \text{ and } S(P,\ f) := \sum_{i=1}^{k} |I_i| \cdot \sup(f[I_i])\ ,$$

$s(P, f) \in \mathbb{R} \cup \{-\infty\}$ and $S(P, f) \in \mathbb{R} \cup \{+\infty\}$ (infima and suprema are taken in $(\mathbb{R}^*, <)$), are called the *lower* and the *upper sum (for P and f)*, respectively. It is easy to see that f is unbounded from above if and only if every upper sum $S(P, f) = +\infty$, and that f is unbounded from below if and only if every lower sum $s(P, f) = -\infty$. We leave the following inequalities for these sums as an exercise.

Proposition 1 (monotonicity of l. and u. sums) *Let $P \subset Q$ be partitions of the interval $[a, b]$ and let $f\colon [a, b] \to \mathbb{R}$. Then*

$$s(P,\ f) \le s(Q,\ f) \textit{ and } S(P,\ f) \ge S(Q,\ f)\ .$$

We prove equivalence of the fourth definition of the Riemann integral.

Proposition 2 (4th definition of the R. $\int$) *Let $a < b$ be real numbers and $f\colon [a, b] \to \mathbb{R}$. Then*

$$f \in \mathrm{R}(a,\, b) \iff \exists\, c\, \forall \varepsilon\, \exists P\, \forall \bar{t}\colon |c - R(P,\, \bar{t},\, f)| < \varepsilon\ .$$

Proof. The implication of $\Rightarrow$ is trivial from the definition of (R) $\int$, because we set $c := \int_a^b f$. We prove $\Leftarrow$. It is easy to see (see Proposition 8 in the last lecture) that f is bounded. Let $d > 0$ be a bounding constant. We take by the hypothesis of the implication the number $c \in \mathbb{R}$ and for the given ε we take the partition $P = (a_0, \dots, a_k)$ of the interval $[a, b]$ with the stated property. We show that for every partition Q of the interval $[a, b]$ with a small norm $\Delta(Q)$ for any test points $\overline{u}$ of Q also $R(Q, \overline{u}, f)$ differs slightly from c. From this it is clear that $f \in \mathrm{R}(a, b)$ and that $c = \int_a^b f$.

So let $Q = (b_0, \dots, b_l)$ and $\overline{u}$ be test points of Q. We can assume that $\Delta(Q) < \frac{1}{2}\min_{1 \le i \le k}(a_i - a_{i-1})$. For each $i = 1, 2, \dots, k$ we define $t_i := u_j$ for (some) u_j minimizing the values

$$\{f(u_j) \mid [b_{j-1},\, b_j] \cap [a_{i-1},\, a_i] \neq \emptyset\}\ .$$

Let X be the set of those intervals $[b_{j-1}, b_j]$ that the interval (b_{j-1}, b_j) contains (necessarily only one) point from P. Then

$$R(Q,\, \overline{u},\, f) + \sum_{[b_{j-1},\, b_j] \in X} (b_j - b_{j-1}) f(u_j) \ge R(P,\, \bar{t},\, f) > c - \varepsilon\ ,$$

because all intervals $[a_{i-1}, a_i]$ are simultaneously covered by intervals $[b_{j-1}, b_j]$ so that each $[b_{j-1}, b_j]$ is used once, except for intervals in X, which are used twice. So

$$R(Q,\, \overline{u},\, f) > c - \varepsilon - l\Delta(Q)d\ .$$

Similarly (by choosing maximizing u_j in $t_i := u_j$ and subtracting the sum over $[b_{j-1},\, b_j] \in X$) we see that also $R(Q, \overline{u}, f) < c + \varepsilon + l\Delta(Q)d$. So

$$|R(Q,\, \overline{u},\, f) - c| < \varepsilon + l\Delta(Q)d\ ,$$

as we promised. $\square$

Let $a < b$ be real numbers and let $\mathcal{D} = \mathcal{D}(a, b)$ denote the set of all partitions of the interval $[a, b]$. Then

$$\underline{\int_a^b} f := \sup(\{s(P,\, f) \mid P \in \mathcal{D}\}) \in \mathbb{R}^*$$

and

$$\overline{\int_a^b} f := \inf(\{S(P,\, f) \mid P \in \mathcal{D}\}) \in \mathbb{R}^*$$

(infima and suprema are again taken in $(\mathbb{R}^*, <)$) is the so-called *lower* and *upper integral (of f over $[a, b]$), respectively.*

Proposition 3 ($\underline{\int} \le \overline{\int}$) *Let $f\colon [a, b] \to \mathbb{R}$ be a function. Then for every two partitions $P, Q \in \mathcal{D}(a, b)$,*

$$s(P, f) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le S(Q, f)\ .$$

Proof. Let f be as stated, and let P and Q be partitions of the interval $[a, b]$. We already know the trick with $R := P \cup Q$. Then $P, Q \subset R$ and, by Proposition 1,

$$s(P,\, f) \le s(R,\, f) \le S(R,\, f) \le S(Q,\, f) \text{ a } s(P,\, f) \le S(Q,\, f)\ .$$

We now use the fact that in any linear order $(X, \prec)$, for every two sets $A, B \subset X$ with $A \preceq B$ we have $\sup(A) \preceq \inf(B)$, if these elements exist. Every $a \in A$ is a lower bound of the set B, so $A \preceq \{\inf(B)\}$. Thus $\inf(B)$ is an upper bound of the set A and $\sup(A) \preceq \inf(B)$. □

Proposition 4 (Riemann = Darboux) *A function f from $[a,b]$ to $\mathbb{R}$ is*

$$f \in \mathrm{R}(a,\, b) \iff \underline{\int_a^b} f = \overline{\int_a^b} f \in \mathbb{R}\ .$$

In the positive case, $(\mathrm{R}) \int_a^b f = \underline{\int_a^b} f = \overline{\int_a^b} f$.

Proof. The implication $\Rightarrow$. Let $f \in \mathrm{R}(a,b)$. Then f is bounded and infima in $s(P,f)$ and suprema in $S(P,f)$ are finite. We can thus approximate them arbitrarily closely by functional values and get that for every ε and every $P \in \mathcal{D}(a,b)$ there are test points $\bar{t}$ of P such that

$$|s(P,\, f) - R(P,\, \bar{t},\, f)| < \varepsilon\ ,$$

and that for every ε and every $P \in \mathcal{D}(a,b)$ there are test points $\bar{t}$ such that

$$|S(P,\, f) - R(P,\, \bar{t},\, f)| < \varepsilon\ .$$

Hence, by Proposition 3 here and Definition 1 in the last lecture, the implication and the last part of the statement follow.

The implication $\Leftarrow$. Let $I := \underline{\int_a^b} f = \overline{\int_a^b} f \in \mathbb{R}$, so f is bounded, and let an ε be given. By this assumption and by Proposition 3 we take $P, Q \in \mathcal{D}(a,b)$ such that $s(P,f) \le I \le S(Q,f)$ and $0 \le S(Q,f) - s(P,f) < \varepsilon$. We put $R := P \cup Q$. By Propositions 1 and 3,

$$s(P,\, f) \le s(R,\, f) \le I,\ R(R,\, \bar{t},\, f) \le S(R,\, f) \le S(Q,\, f)$$

for all test points $\bar{t}$ in R. Thus also $|R(R, \bar{t}, f) - I| < \varepsilon$ and $f \in \mathrm{R}(a, b)$ by Proposition 2. $\square$

• *The Henstock–Kurzweil integral — the correct definition of the Riemann integral.* Last time we saw that (N) $\int_0^1 1/\sqrt{x} = 2$, but that (R) $\int_0^1 1/\sqrt{x}$ does not exist, because the integrand $1/\sqrt{x}$ is unbounded. The inability of the Riemann integral to integrate unbounded functions is its serious shortcoming. In 1957 the Czech mathematician *Jaroslav Kurzweil (1926–2022)* and a little later the English mathematician *Ralph Henstock (1923–2007)* modified the condition $\Delta(P) < \delta$ and improved the Riemann integral to be able to integrate unbounded functions. We present the definition of their integral and prove one basic theorem about it. The necessary change in the definition in the last lecture is optically small but substantial.

Let $I \subset \mathbb{R}$ be an interval. We call each function $\delta_c\colon I \to (0, +\infty)$ a *gauge (on I)*. A partition $P = (a_0, a_1, \ldots, a_k)$ of the interval $[a, b]$ and its test points $\bar{t} = (t_1, \ldots, t_k)$, $t_i \in [a_{i-1}, a_i]$, are δ_c*-fine* if

$$\forall\, i = 1,\, 2,\, \ldots,\, k\colon a_i - a_{i-1} < \delta_c(t_i)\ .$$

For example, if $\Delta(P) < \delta$, then the partition P together with any test points $\bar{t}$ are δ_c-fine for the constant gauge $\delta_c = \delta$.

Proposition 5 (Cousin's Lemma) *Let $a < b$ be in $\mathbb{R}$. For every gauge $\delta_c \colon [a, b] \to (0, +\infty)$ there exist δ_c-fine partition $P \in \mathcal{D}(a, b)$ with test points $\bar{t}$. Even every finite system $[a_i, b_i]$, $i \in I$, of mutually disjoint subintervals $[a_i, b_i] \subset [a, b]$ with test points $t_i \in [a_i, b_i]$, for which $b_i - a_i < \delta_c(t_i)$ for $\forall\, i \in I$, can be completed to a δ_c-fine partition of $[a, b]$ with test points $\bar{t}$.*

Proof. (abridged) The set

$$M := [a,\, b] \setminus \bigcup_{i \in I} (a_i,\, b_i)$$

is compact and therefore we can select from its (open) cover

$$M \subset \bigcup_{x \in M} U(x,\, \delta_c(x)/2)$$

a finite subcover $U(x_i, \delta_c(x_i)/2)$, $i = 1, 2, \ldots, n$. We add to the intervals $[a_i, b_i]$, $i \in I$, suitable closed subintervals of the intervals $(x_i - \delta_c(x_i), x_i + \delta_c(x_i))$ (containing the corresponding point x_i) and obtain a partition of $[a, b]$. The obtained test points $\bar{t}$ are the t_i, $i \in I$, and $x_1, \ldots, x_n$. The result is δ_c-fine. □

The definition of the Henstock–Kurzweil integral follows. The previous proposition shows that the implication in it can always be satisfied non-vacuously, by a valid assumption. The definition is therefore correct and does not allow logical fallacy like computing the limit of a function at a point which is not the limit point of the definition domain.

Definition 6 (Henstock–Kurzweil integral) *A function $f\colon [a,b] \to \mathbb{R}$ is Henstock–Kurzweil integrable, symbolically written $f \in \mathrm{HK}(a,b)$, if there is a number $L \in \mathbb{R}$ such that for $\forall\,\varepsilon\ \exists\,\delta_c$, where δ_c is a gauge on $[a,b]$, such that for every partition P of $[a,b]$ and test points $\bar{t}$ of P it holds that*

$$P \text{ and } \bar{t} \text{ are } \delta_c\text{-fine} \Rightarrow |R(P, \bar{t}, f) - L| < \varepsilon\ .$$

Then we also write

$$(\mathrm{HK})\int_a^b f = L \quad \text{or} \quad (\mathrm{HK})\int_a^b f(x)\,\mathrm{d}x = L$$

and say that the Henstock–Kurzweil integral of the function f over the interval $[a,b]$ equals L.

It is clear from the definition that $\mathrm{R}(a,b) \subset \mathrm{HK}(a,b)$.

The following theorem[1] shows that the Henstock–Kurzweil integral is finally the right partner for the Newton integral.

Theorem 7 (HK. $\int$ and N. $\int$) *Let $a < b$ be in $\mathbb{R}$, $F\colon [a,b] \to \mathbb{R}$ be a continuous function and let $F' = f$ on (a,b) (the values $f(a)$ and $f(b)$ are arbitrary). Then $f \in \mathrm{HK}(a,b)$ and*

$$(\mathrm{HK})\int_a^b f = F(b) - F(a) = (\mathrm{N})\int_a^b f\ .$$

Proof. Let ε and $x \in (a,b)$ be given. Due to the equality $F'(x) =$

[1]Together with proof it is taken from J. Lukeš and J. Malý, *Measure and integral*, matfyzpress, Praha 2013, pp. 96–97.

$f(x)$ there is a value $\delta_c(x) > 0$ such that for every $y \in [a,\, b]$,

$$y \in U(x,\, \delta_c(x)) \Rightarrow |F(y) - F(x) - f(x)(y-x)| \le \varepsilon |y-x| \;. \quad (*)$$

Moreover, there exist values $\delta_c(a) > 0$ and $\delta_c(b) > 0$ such that $|f(a)\delta_c(a)|$, $|f(b)\delta_c(b))| < \varepsilon$ and that $|F(y) - F(a)|$, $|F(z) - F(b)| < \varepsilon$ for every $y \in [a, a + \delta_c(a))$ and every $z \in (b - \delta_c(b), b]$. If the partition $P = (a_0, \ldots, a_k) \in \mathcal{D}(a,b)$ with test points $\overline{t}$ are δ_c -fine, then for every test point in an interval $t_i \in [a_{i-1}, a_i]$, with $t_i \neq a, b$, one has that

$$\begin{aligned}
&|F(a_i) - F(a_{i-1}) - f(t_i)(a_i - a_{i-1})| \overset{\Delta\text{ -ineq.}}{\le} |F(a_i) - F(t_i) - \\
&- f(t_i)(a_i - t_i)| + |F(t_i) - F(a_{i-1}) - f(t_i)(t_i - a_{i-1})| \\
&\overset{(*)}{\le} \varepsilon |a_i - t_i| + \varepsilon |t_i - a_{i-1}| = \varepsilon (a_i - a_{i-1}) \;.
\end{aligned}$$

If $t_i \in [a_{i-1}, a_i]$ and $t_i \in \{a, b\}$, then

$$\begin{aligned}
&|F(a_i) - F(a_{i-1}) - f(t_i)(a_i - a_{i-1})| \\
&\overset{\Delta\text{ -ineq.}}{\le} |F(a_i) - F(a_{i-1})| + |f(t_i)(a_i - a_{i-1})| \overset{\text{Moreover, there }\ldots}{<} 2\varepsilon
\end{aligned}$$

because $i = 1$ and $t_1 = a$ or $i = k$ and $t_k = b$. According to these two estimates, we have that

$$\begin{aligned}
&|F(b) - F(a) - R(P,\, \overline{t},\, f)| \overset{\Delta\text{ -ineq.}}{\le} \\
&\le \sum_{i=1}^{k} |F(a_i) - F(a_{i-1}) - (a_i - a_{i-1}) f(t_i)| < \varepsilon (b-a) + 4\varepsilon
\end{aligned}$$

so that $F(b) - F(a) = \text{(HK)} \int_a^b f$. $\square$

Corollary 8 *Let* $1/\sqrt{0} := 1$. *Then* (HK) $\int_0^1 1/\sqrt{x} = 2$.

• *Integration by parts and by substitution for* (R) $\int_a^b f$. We present the third version of these two integration formulae. The first one was for primitive functions, the second one for the Newton integral, and this one is for the Riemann integral. Substitution now turns out to be surprisingly non-trivial. In the following theorem, the values $f(a)$, $f(b)$, $g(a)$ and $g(b)$ are arbitrary.

Theorem 9 (integration by parts for R. $\int$) *Let* $a < b$ *be in* $\mathbb{R}$, *let the functions* $F, G, f, g\colon (a, b) \to \mathbb{R}$ *satisfy on* (a, b) *that* $F' = f$ *and* $G' = g$, *and let* $Fg, fG \in \mathrm{R}(a, b)$. *Then the equality holds that*

$$\int_a^b Fg = \big[FG\big]_a^b - \int_a^b fG\ .$$

Proof. By the linearity of the Riemann integral also $fG + Fg \in \mathrm{R}(a, b)$. From this linearity, from $(FG)' = fG + Fg$ on (a, b) and from FTC 2 (Theorem 15 of Lecture 12) we have that

$$\begin{aligned}
(\mathrm{R})\int_a^b fG + (\mathrm{R})\int_a^b Fg \ &= \ (\mathrm{R})\int_a^b (fG + Fg) \\
&= \ (\mathrm{N})\int_a^b (fG + Fg) = \big[FG\big]_a^b\ ,
\end{aligned}$$

which is a rearrangement of the stated equality. $\square$

A simple but not completely satisfactory formula for Riemann integration by substitution is this.

Theorem 10 (R. $\int$ by substitution) *Let $G\colon [a,b] \to \mathbb{R}$ have on $[a,b]$ continuous derivative G' and let $f\colon G[\,[a,b]\,] \to \mathbb{R}$ be continuous. Then the equality of Riemann integrals holds that*

$$\int_{G(a)}^{G(b)} f = \int_a^b f(G)G'\ .$$

Proof. For $x \in G[\,[a,b]\,]$ we consider the function

$$F(x) := \int_{G(a)}^{x} f$$

(by the results of the last lecture it is well defined). According to FTC 1 (Theorem 16 of the last lecture) and derivatives of composite functions, the function $F(G)$ is on $[a,b]$ a primitive function of $f(G)G'$. By FTC 2 (Theorem 15 of the last lecture) and definition of the function F we have that $(F(G(a)) = 0)$

$$\int_a^b f(G)G' = \big[F(G)\big]_a^b = F(G(b)) - F(G(a)) = \int_{G(a)}^{G(b)} f\ .$$

$\square$

Unsatisfactory, however, is that here we are working only with Newton integrals, and that this theorem is already contained in part 1 of Theorem 5 on substitution in the Newton integral in Lecture 11. A theorem on substitution directly for the Riemann integral was proven by H. Kestelman only in 1961. We present here an improved version with equivalence for Riemann integrability, due to the Czech mathematicians D. Preiss and J. Uher in 1970[2].

[2] Poznámka k větě o substituci pro Riemannův integrál, *Časopis pěst. mat.* **95** (1970), 345–347.

Theorem 11 (D. Preiss and J. Uher, 1970) *Let $g \in \mathrm{R}(a,b)$, for $x \in [a,b]$ let $G(x) := \int_a^x g$ and let $f\colon G[\,[a,b]\,] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on the interval $G[\,[a,b]\,]$ if and only if $f(G)g \in \mathrm{R}(a,b)$, and in the positive case the equality of Riemann integrals holds that*

$$\int_{G(a)}^{G(b)} f = \int_a^b f(G)g\ .$$

For proofs see the original article `https://eudml.org/doc/19168` or the recent `https://arxiv.org/abs/1105.5938` and `https://arxiv.org/abs/1904.07446`.

• *Use of integrals in formulas for lengths, areas and volumes.* We denote by the symbol $|uv|$ (always ≥ 0) the length of the straight segment with endpoints $u, v \in \mathbb{R}^2$.

Definition 12 (length of G_f) *We say that $f\colon [a,b] \to \mathbb{R}$ has rectifiable graph if the supremum*

$$\ell(f) := \sup\big(\{\textstyle\sum_{i=1}^k \big|\big(a_{i-1}, f(a_{i-1})\big)\,\big(a_i, f(a_i)\big)\big| \mid$$
$$\mid (a_0,\,\dots,\,a_k) \in \mathcal{D}(a,\,b)\}\big)$$

is finite. The number $\ell(f)$ is then called the length of the graph of the function f.

This supremum is actually the supremum of lengths of broken lines inscribed in the graph G_f of f.

Using the following formula for $\ell(f)$, we are able to calculate, for example, perimeters of rectangles and can match by this results of the elementary school mathematics. We just divide the perimeter into four sides and rotate the vertical ones by $\pi/2$ (or just by ε) to get four graphs of functions.

Theorem 13 (length of G_f) *Suppose that $f\colon [a,b] \to \mathbb{R}$ is a continuous function that has on (a,b) finite derivative $f' \in \mathrm{R}(a,b)$. Then f has a rectifiable graph with length*

$$\ell(f) = \int_a^b \sqrt{1+(f')^2}\ .$$

Proof. Let $g := \sqrt{1+(f')^2}$. By the results of the last lecture the Riemann integral $\int_a^b g$ exists. The sum in Definition 12, which we denote as $K(P, f)$, does not decrease under subdivision of $P = (a_0, \dots, a_k)$, so for any sequence $(P_n) \subset \mathcal{D}(a,b)$ with $\lim \Delta(P_n) = 0$ one has that

$$\lim\ K(P_n,\ f) = \ell(f)$$

or, for a non-rectifiable graph, this limit is always $+\infty$. But

$$K(P,\ f) = \sum_{i=1}^{k}(a_i - a_{i-1})\sqrt{1+[(f(a_i)-f(a_{i-1}))/(a_i-a_{i-1})]^2}$$

and by the Lagrange mean value theorem,

$$\frac{f(a_i)-f(a_{i-1})}{a_i-a_{i-1}} = f'(t_i)$$

for some $t_i \in (a_{i-1}, a_i)$. Let us denote these test points as $\overline{t}$. So for (P_n) as above,

$$\int_a^b g = \lim_{n\to\infty} R(P_n,\ \overline{t(n)},\ g) = \lim_{n\to\infty} K(P_n, f) = \ell(f)\ .$$

$\square$

This formula can be extended to curves of the form $\varphi\colon [a,b] \to \mathbb{R}^n$.

We did not define areas of planar regions independently of the integral, the same for the volume in $\mathbb{R}^3$, and so the following two formulas are — at least in our our lectures — unlike the length of the graph only at the level of definitions.

Definition 14 (area between two graphs) *Let $f, g \in \mathrm{R}(a,b)$ and $f \leq g$ on $[a,b]$. Then*

$$\begin{aligned} &\mathit{area}\big(\{(x,\, y) \in \mathbb{R}^2 \mid x \in [a,\, b] \wedge f(x) \leq y \leq g(x)\}\big) \\ &:= \int_a^b (g - f)\ . \end{aligned}$$

For any non-negative function $f\colon [a,b] \to \mathbb{R}$ we define *the solid of revolution (obtained by rotating G_f around the axis x)* as

$$V(a,\, b,\, f) := \{(x,\, y,\, z) \in \mathbb{R}^3 \mid x \in [a,\, b] \wedge y^2 + z^2 \leq f(x)^2\}\ .$$

Definition 15 (solid of revolution) *Let $f \in \mathrm{R}(a,b)$ be nonnegative. Then*

$$\mathit{volume}\big(V(a,\, b,\, f)\big) := \pi \int_a^b f^2\ .$$

Intuitively — or as a mnemonic — the Riemann integral

$$\int_a^b \pi \cdot f(x)^2 \, \mathrm{d}x$$

for the volume of the body $V(a,b,f)$ follows from the formula πr^2 for the area of the circle with radius $r > 0$. For x running in $[a,b]$

the integral adds the volumes

$$\pi \cdot f(x)^2 \, \mathrm{d}x$$

of thin pancakes with radii $f(x)$ and thickness $\mathrm{d}x$. As an exercise, derive a formula for the surface, that is, the surface area, of the solids $V(a,b,f)$. By it one can calculate the surface of the sphere $V(-r,r,\sqrt{r^2-x^2})$ with radius $r>0$.

• *Estimates of sums using integrals.* They are useful, for example, in analytic number theory, where sums of the form $\sum_{n\in X} f(n)$, for sets $X\subset\mathbb{Z}$ and functions $f(x)$ given by analytic formulas, appear frequently. We start with an estimate that is simple but has a wide scope.

Proposition 16 ($\sum f(n)$ for monotone f) *Let $a<b$ be integers and $f\colon [a,b]\to\mathbb{R}$ be a monotone function. Then*

$$\sum_{a<n\le b} f(n) = (\mathrm{R})\int_a^b f + \theta(f(b)-f(a)) \; ,$$

for some number $\theta\in[0,1]$.

Proof. The integral exists due to the monotonicity of the function f (Theorem 14 in the previous lecture). We assume that f is non-decreasing, the case of non-increasing f is solved similarly. So now we prove the inequalities

$$0 \le \sum_{a<n\le b} f(n) - \int_a^b f \le f(b)-f(a) \; .$$

For $b=a+1$ they hold: the sum is $f(a+1)$ and because $f(a)\le f(x)\le f(a+1)$ for $x\in[a,a+1]$, by the monotonicity of $(\mathrm{R})\int$

(which we probably did not treat explicitly) one has that $f(a) = f(a) \cdot 1 \le \int_a^{a+1} f \le f(a+1) \cdot 1 = f(a+1)$. Adding these simple inequalities with the limits $a = m, b = m+1$ for $m = a, a+1, \dots, b-1$ we get the general case. $\square$

For example, for the harmonic numbers $H_n := \sum_{i=1}^n 1/i$ we get the estimate that for $n \ge 3$,

$$\begin{aligned} H_n &= 1 + \sum_{i=2}^n \frac{1}{i} = 1 + \int_1^n 1/x + \theta(1/n - 1) = [\log x]_1^n + \delta \\ &= \log n + \delta, \quad 1/n \le \delta \le 1 \,. \end{aligned}$$

Corollary 17 (integral criterion) *Suppose that $m \in \mathbb{N}$ and that $f\colon [m, +\infty) \to \mathbb{R}$ is a non-negative and non-increasing function. Then the series*

$$\sum_{n=m}^{\infty} f(n) \text{ converges} \iff \lim_{n \to \infty} \int_m^n f < +\infty \,.$$

Proof. As we know, these Riemann integrals exist due to the monotonicity of f. According to the previous proposition, for each integer $N \ge m+2$ we have the identity

$$\sum_{n=m}^{N} f(n) = f(m) + \int_m^N f + \underbrace{\theta(f(N) - f(m))}_{\in[-f(m),0]} \,.$$

Hence the stated equivalence for $N \to \infty$. $\square$

For example, the series $\sum_{n=2}^{\infty} 1/n \log n$ diverges, i.e., has the sum $+\infty$, because

$$\lim_{n \to \infty} \int_2^n \frac{\mathrm{d}y}{y \log y} = \lim_{n \to \infty} \big[\log(\log y)\big]_2^n = +\infty \,.$$

Conversely, we prove convergence of the series $\sum_{n=2}^{\infty} 1/n(\log n)^c$ for every real $c > 1$ by the same method.

We present a variant of Proposition 16 for functions with integrable derivative; then a more accurate estimate of the sum in the form of an identity is obtained. Recall that $\lfloor a \rfloor$ is the lower integer part of $a \in \mathbb{R}$, the largest $m \in \mathbb{Z}$ with $m \le a$. We introduce the notation $\langle a \rangle := a - \lfloor a \rfloor - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2})$.

Theorem 18 ($\sum f(n)$ for differentiable f) *Let $a < b$ be real numbers and let $f \in \mathrm{R}(a,b)$ have on (a,b) the derivative $f' \in \mathrm{R}(a,b)$. Then the formula holds that*

$$\sum_{a<n\le b} f(n) = \int_a^b f + \underbrace{\int_a^b \langle x \rangle f'(x)}_{T} - \big[\langle x \rangle f(x)\big]_a^b \,.$$

Proof. [3] The formula is additive in intervals $[a,b)$, so it is enough to consider only the case that $m \le a < b \le m+1$ for some $m \in \mathbb{Z}$. Integration by parts (Theorem 9) then gives that

$$T = \int_a^b (x-m-1/2) f'(x) = \big[(x-m-1/2) f(x)\big]_a^b - \int_a^b f \,.$$

We substitute this in the right-hand side of the formula and see that only $(\lfloor b \rfloor - m) f(b)$ remains of it. For $b < m+1$ it is 0, which agrees with the left-hand side. For $b = m+1$ it is $f(m+1)$, again in agreement with the left-hand side. $\square$

[3] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford 1986, pp. 13–14.

For harmonic numbers, the more accurate estimate

$$H_n = \sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + O(1/n) \ \ (n \in \mathbb{N}) \ ,$$

which we mentioned in part 1 of Theorem 4 of Lecture 4, is easily derived with this formula.

We conclude the lecture and the whole course with Abel's summation formula. For a sequence $(a_n) = (a_1, a_2, \dots) \subset \mathbb{R}$ and a number $x \in \mathbb{R}$ we define

$$A(x) := \sum_{n \le x} a_n \ ,$$

with an empty sum defined as 0.

Theorem 19 (Abel's[a] summation) *Let $(a_n) \subset \mathbb{R}$, $a < b$ be positive real numbers and $f\colon [a,b] \to \mathbb{R}$ be a function that has on (a,b) derivative $f' \in \mathrm{R}(a,b)$. Then*

$$\sum_{a<n\le b} a_n f(n) = \big[A(x)f(x)\big]_a^b - \underbrace{\int_a^b A(x)f'(x)}_{T} \ .$$

[a]Named after the Norwegian mathematician Niels Henrik Abel (1802–1829).

Proof. We use Titchmarsh's trick from the previous proof. The formula is again additive in intervals $[a,b)$, so again it is enough to consider only the case that $m \le a < b \le m+1$ for some $m \in \mathbb{N}_0$. FTC 2 (Theorem 15 in the last lecture) then gives that

$$T = \int_a^b A(m)f'(x)\,\mathrm{d}x = A(m)\big[f(x)\big]_a^b \ .$$

We substitute it in the right-hand side of the formula and see that it turns in $(A(b) - A(m))f(b)$. For $b < m + 1$ it is 0, in agreement with the left-hand side. For $b = m + 1$ it is $a_{m+1}f(m + 1)$, again in agreement with the left-hand side. $\square$

THANK YOU FOR YOUR ATTENTION!